Vtree Alignment and Higher-Order Four-Corner Defects in Exact Boolean Factorization

Research draft

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Abstract

We isolate two structural limits in exact AND-factor extraction from vtree-structured Boolean representations. First, maximizing the number of factors exposed by exact vtree cuts is a complete objective: its global optimum equals the semantic prime-factor count, while hard carrier-subtree constraints are compatible with semantic prime alignment exactly when the carrier and semantic block families are laminar. Second, a nonzero four-corner defect with varying supports U and W proves that some semantic prime block meets both supports, but this cannot in general be reduced to a pairwise interaction. The family supported only on 0^n and 1^n is semantically prime although every fixed-context two-variable slice has rank at most one.

1 Setup

Let Π_f denote the canonical finest partition of the variable set for which a nonfalse Boolean function factors as a conjunction of functions on disjoint blocks. For a vtree T, let $E_T(f)$ be the finest partition exposed recursively by the exact compressed-SDD cut criterion.

Theorem 1 (Exposure optimum)**.** *For every nonfalse Boolean function f,*

$$\max_T |E_T(f)| = |\Pi_f|.$$

Proof. Every exposed factorization is valid, hence is a coarsening of Π_f. Conversely, build a binary vtree with one subtree for each block of Π_f and combine those subtrees arbitrarily. Every internal split is a union of prime blocks against its complement and therefore factors; recursion reaches all prime blocks. □

Theorem 2 (Carrier/semantic compatibility)**.** *Let* Π *and* Σ *be partitions of the same ground set. A binary rooted leaf-labelled tree can contain every block of both partitions as a subtree leaf set if and only if the combined block family* $\Pi \cup \Sigma$ *is laminar.*

Proof. Subtree leaf sets form a laminar family, giving necessity. Conversely, represent the inclusion poset of a finite laminar family as a rooted tree and refine nodes of degree greater than two by inserting binary nodes. □

2 Defect support

Consider a cut $A \mid B$ and a nonzero four-corner determinant witnessed by row assignments a_0, a_1 and column assignments b_0, b_1. Let $U \subseteq A$ and $W \subseteq B$ be the coordinates on which the two row and two column assignments differ.

Lemma 1 (Support crossing)**.** *Some block of* Π_f *intersects both* U *and* W*.*

Proof. If not, after fixing all nonvarying coordinates every prime factor depends on row variables only, column variables only, or neither. Its 2×2 corner table is rank one. Entrywise multiplication of rank-one outer products remains rank one, contradicting the nonzero determinant. □

Corollary 1. *If* $U = \{x\}$ *and* $W = \{y\}$*, then* x *and* y *are in the same semantic prime block.*

Theorem 3 (No universal pairwise localization)**.** *For* $n \geq 3$*, let* $f_n(x) = 1$ *exactly for* $x = 0^n$ *and* $x = 1^n$*. Then* Π_{f_n} *consists of one block, but every two-variable restriction obtained by fixing all remaining coordinates has matrix rank at most one. For every nontrivial cut* $A \mid B$*, a four-corner defect exists only when the witness changes all coordinates of* A *and all coordinates of* B*.*

Proof. A nontrivial product decomposition would make the two-point satisfying relation a nontrivial Cartesian product, which would contain mixed combinations; hence the function is prime. A fixed-context two-variable slice contains at most one of $0^n, 1^n$, so it has at most one nonzero entry. Across $A \mid B$, the only two nonzero matrix entries are the all-zero and all-one row/column pairs; a nonzero 2×2 minor must select both, forcing all cut-side coordinates to change. □

3 Interpretation

The results support hyperedge-valued defect evidence rather than automatic pairwise must-link edges. They also show that carrier blocks should not be hard-wired into vtrees when they cross emerging semantic factors.