

Observation Topologies for Finite Propositional Semantics: Distinguishability, Distance, and Inference

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Abstract

Fix a propositional language over finitely many variables and regard its truth assignments as points. A selected finite family of formulas acts as a family of positive-certification tests, and the topology generated by their truth regions records the distinctions supported by those tests. In this finite setting, observation signatures give an explicit coordinate description of the resulting Alexandrov topology: topological indistinguishability is equality of signatures, the specialization preorder is inclusion of positive observations, and a weighted directed set-difference quasi-pseudometric generates the same topology. Symmetrizing that directed distance gives weighted Hamming distance. We distinguish positive observation from a complete two-sided record, and observation refinement from premise restriction. Classical inference itself remains ordinary inclusion of the common premise region in the conclusion region. The paper is a self-contained expository synthesis of standard finite Boolean, order-topological, observational, and quasi-metric constructions; its aim is to make their interfaces and presentation-dependence explicit in one propositional model rather than to claim a new class of topological spaces.

Keywords: finite topology; Alexandrov space; propositional logic; observation; specialization preorder; quasi-pseudometric; Hamming distance.

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1 Introduction

Propositional formulas can be represented by the truth assignments on which they hold. If every such subset of a finite valuation space is declared open, the resulting topology is discrete and contains no structure beyond classical truth-set semantics. A nontrivial topological treatment begins instead with a restricted family of formulas that can be used as observations. Two truth assignments may then be logically different but observationally indistinguishable.

This paper develops five linked ideas:

- (i) complete truth assignments give the atomic partition for finite Boolean semantics;
- (ii) selected observable formulas determine distinguishability and indistinguishability;
- (iii) the specialization preorder records asymmetric logical separation;
- (iv) XOR of observation signatures gives symmetric disagreement, while a source mask gives directed loss; and
- (v) adding observations refines the topology, while adding premises restricts the current space of possibilities.

The intended reader knows propositional truth tables and basic set notation but may be meeting finite topology, specialization order, or topological semantics for the first time. The expository aim is to use one finite observation table to show which constructions share data and which do not: in particular, truth-region intersection versus intersection of topologies, complete semantic profiles versus partial certificates, and an indexed observation presentation versus the topology it generates.

The numbered propositions and theorems below are included to make the synthesis self-contained; they are not priority claims for the underlying finite-topology, order, Hamming, or quasi-metric facts. The contribution is organizational: the same finite signature data are used consistently to compare positive evidence, two-sided observation, refinement, update, distance, and inference.

The qualitative construction can be summarized as

$$\Phi \longmapsto o_\Phi \longmapsto \tau_\Phi \longleftrightarrow \preceq_\Phi .$$

Given positive weights $\mathbf{a} = (a_1, \dots, a_m)$, the same presentation also gives

$$(\Phi, \mathbf{a}) \longmapsto d_{\Phi, \mathbf{a}}^+ \longmapsto (\preceq_{\Phi}, \tau_{\Phi}).$$

Here Φ is an indexed family of observable formulas, o_{Φ} is the observation signature, τ_{Φ} is the generated topology, $\preceq_{\Phi}$ is its specialization preorder, and d_{Φ}^+ is a directed logical distance. The last arrow is not reversible: the topology determines its specialization preorder on this finite carrier, but it does not determine the chosen weights or observation presentation.

The Boolean representation underlying the truth-region map is classical and is closely related to the representation of Boolean algebras by fields of sets [15]. Topological semantics for logical operations has a long history, notably in the work of McKinsey and Tarski [11]. Finite topologies and their order-theoretic structure are classical as well [16, pp. 325–327]; see also [3]. Sierpiński-valued predicates provide a standard description of opens and positive observation [17, Ch. 1]; see also [18, Example 1.12(3)]. Observation frames provide a related abstract separation between observable predicates and all predicates [4]. Abramsky and Vickers place observation and testing in a broader algebraic semantics of observational logic [1]. More recently, Achilleos and Kyriakou study topologies generated by finite behavioural observations and characterize open sets as properties verifiable by those observations [2]. Their process-semantic setting is different from the finite propositional carrier used here, but it is a close conceptual precedent for the observation-to-topology viewpoint. The purpose here is to assemble these ideas around partitioned propositional states, asymmetric distinguishability, signature distance, refinement, and inference. No new duality, completeness theorem, or general existence theorem for observation topologies is claimed.

Two further precedents use essentially the same object–attribute data. In Pawlak’s information systems, equality of attribute descriptions induces an indiscernibility partition, whose lower and upper approximations are interior and closure in the associated partition topology [12, Sec. 2.2, pp. 343–344]. That construction agrees here with two-sided observation; positive observation may instead retain strict specialization comparisons. In formal concept analysis, the incidence $v \text{ I } i$ defined by $v \models \varphi_i$ is a formal context, and a finite conjunction of attributes has the corresponding extent [8, Sec. 1.1]. Topological constructions have also been built directly from formal contexts; Pei et al. construct a topology on the attribute set rather than on the valuation/object carrier used here [14]. Our topology also admits arbitrary unions of such extents, which need not themselves be extents, so its open-set lattice is not in general the concept lattice. For $\Phi = (X, Y)$, for instance, the three-element set of valuations $\{(1, 0), (1, 1), (0, 1)\}$ is open but is not an extent of the context with just the attributes X and Y .

Table 1 records the distinctions that organize the paper. In particular, changing what can be observed and changing which valuations remain possible are different operations.

2 Valuations, truth regions, and partitions

Let $P = \{p_1, \dots, p_n\}$ be a finite set of propositional variables and let

$$\Omega_P = \{0, 1\}^P$$

be the set of valuations. For $v \in \Omega_P$, the expression $\mathcal{V}_v(\mathcal{L})$ is the value of a formula $\mathcal{L}$ at v , under the usual classical truth tables. We identify T with 1 and F with 0.

Operation	Observation data	Carrier	Effect
Positive observation	Φ	Ω_P	Generates τ_Φ from affirmative truth regions.
Two-sided observation	$\Phi^\pm$	Ω_P	Makes complete signature fibers clopen; yields a partition topology.
Observation refinement	enlarge Φ	Ω_P	Refines distinguishability and can only make the topology finer.
Premise update	keep Φ	$S_\Gamma \subseteq \Omega_P$	Restricts possible valuations and uses the induced subspace topology.

Table 1: Four operations that share truth-table data but should not be conflated.

2.1 Formula tuples, truth vectors, and truth regions

Definition 2.1 (Formula tuple and truth vector). A *formula tuple* of arity m is an ordered tuple of formulas

$$\mathbf{L} = (\mathcal{L}_1, \dots, \mathcal{L}_m).$$

At a valuation $v \in \Omega_P$, its *truth vector* is the Boolean vector

$$\mathcal{V}_v(\mathbf{L}) := (\mathcal{V}_v(\mathcal{L}_1), \dots, \mathcal{V}_v(\mathcal{L}_m)) \in \{0, 1\}^m.$$

A formula tuple is therefore syntactic, whereas a truth vector is one of its possible semantic values. Neither should be confused with a valuation $v \in \Omega_P$, which assigns truth values to every variable in the fixed language and serves as a point of the topological carrier.

Definition 2.2 (Truth set or truth region). For a formula $\mathcal{L}$, define

$$\varsigma(\mathcal{L}) := \{v \in \Omega_P : \mathcal{V}_v(\mathcal{L}) = \mathbf{T}\}.$$

The notation $\mathcal{V}_v^T(\mathcal{L})$ abbreviates $v \in \varsigma(\mathcal{L})$, and $\mathcal{V}_v^F(\mathcal{L})$ abbreviates $v \notin \varsigma(\mathcal{L})$.

The terms *truth set* and *truth region* refer to this same subset of Ω_P . We use “region” when emphasizing its role inside a topological space. The superscripts T and F above denote truth assertions; they do not define additional numerical evaluation maps.

Using a fixed carrier ensures that ς depends only on the logical equivalence class of $\mathcal{L}$. It obeys

$$\begin{aligned} \varsigma(\mathcal{L}_1 \wedge \mathcal{L}_2) &= \varsigma(\mathcal{L}_1) \cap \varsigma(\mathcal{L}_2), \\ \varsigma(\mathcal{L}_1 \vee \mathcal{L}_2) &= \varsigma(\mathcal{L}_1) \cup \varsigma(\mathcal{L}_2), \\ \varsigma(\neg \mathcal{L}) &= \Omega_P \setminus \varsigma(\mathcal{L}). \end{aligned}$$

Write $\mathcal{L}_1 \uplus \mathcal{L}_2$ for Boolean exclusive OR. For two formulas, their union separates into an overlap and an exclusive remainder:

$$\varsigma(\mathcal{L}_1 \vee \mathcal{L}_2) = \varsigma(\mathcal{L}_1 \wedge \mathcal{L}_2) \dot{\cup} \varsigma(\mathcal{L}_1 \uplus \mathcal{L}_2).$$

Equivalently,

$$\mathcal{L}_1 \vee \mathcal{L}_2 \equiv (\mathcal{L}_1 \wedge \mathcal{L}_2) \uplus (\mathcal{L}_1 \uplus \mathcal{L}_2),$$

because the overlap and exclusive remainder are disjoint. The complete two-formula truth-pattern partition is

$$\begin{array}{ll} \mathcal{L}_1 \wedge \mathcal{L}_2, & \mathcal{L}_1 \wedge \neg \mathcal{L}_2, \\ \neg \mathcal{L}_1 \wedge \mathcal{L}_2, & \neg \mathcal{L}_1 \wedge \neg \mathcal{L}_2, \end{array}$$

after empty cells are removed. Thus XOR identifies the two exclusive cells; it is not a conversion of the conjunction into a disjunction.

For every $v \in \Omega_P$, define its complete minterm

$$m_v := \bigwedge_{i=1}^n \ell_i, \quad \ell_i = \begin{cases} p_i, & v(p_i) = 1, \\ \neg p_i, & v(p_i) = 0. \end{cases}$$

The family $\{\varsigma(m_v) : v \in \Omega_P\}$ is the singleton partition of Ω_P . Every formula has the canonical complete disjunctive form

$$\mathcal{L} \equiv \bigvee_{v \in \varsigma(\mathcal{L})} m_v,$$

where the empty disjunction is F and the empty conjunction is T. Thus the complete minterms give the atoms of the finite Boolean algebra $\mathcal{P}(\Omega_P)$. They form a topological basis only when the topology is discrete.

For two variables use the cell labels

$$\begin{array}{ll} A = X \wedge \neg Y, & B = X \wedge Y, \\ C = \neg X \wedge Y, & D = \neg X \wedge \neg Y, \end{array} \quad \Omega_{X,Y} = \{A, B, C, D\}.$$

Each letter denotes the unique valuation satisfying the displayed minterm; the same letter may also label that singleton truth cell when the type is clear. For example,

$$\varsigma(X) = \{A, B\}, \quad \varsigma(Y) = \{B, C\}, \quad \varsigma(X \Rightarrow Y) = \{B, C, D\}.$$

3 Observation-generated spaces

The topology should record a limited observational vocabulary rather than all definable subsets.

Definition 3.1 (Observation family and signature). Let $I := \{1, \dots, m\}$ and let the observations be the indexed finite family

$$\Phi = (\varphi_i)_{i \in I}.$$

Put

$$b_i(v) := \mathbf{1}[v \in \varsigma(\varphi_i)]$$

and define the observation signature

$$o_\Phi(v) := (b_i(v))_{i \in I} \in \{0, 1\}^I.$$

When subset notation is used, we identify this vector with its support

$$o_\Phi(v) = \{i \in I : v \models \varphi_i\} \subseteq I.$$

Different indices may carry logically equivalent formulas, or even literal repetitions, and may receive distinct weights.

The truth vector of the indexed formula tuple Φ at v is exactly the observation signature:

$$\mathcal{V}_v(\Phi) = o_\Phi(v).$$

The signature notation emphasizes that these coordinates are the observations available to the topology.

The evidence convention is asymmetric: a subbasic event records positive certification that φ_i is true. Failure to certify φ_i is not itself a positive certificate for $\neg\varphi_i$. If both outcomes of every Boolean test are operationally available, the appropriate family is $\Phi^\pm$ from Proposition 3.5 below, and the resulting topology is the two-sided partition topology. Here “positive” is relative to the selected tests; a selected formula may itself contain a negation.

The signature is a complete semantic profile: $b_i(v) = 0$ asserts that φ_i is false at v . It does not mean that a certificate has merely failed to arrive. The topology describes the affirmative distinctions allowed by the selected interface; it does not by itself model time, partial transcripts, or delivery of certificates. Likewise, the distances in Section 6 are defined on complete profiles and are not automatically measurable from an unfinished positive-only transcript.

For a family $\mathcal{G}$ of subsets of Ω_P , let $\text{Top}(\mathcal{G})$ denote the smallest topology on Ω_P containing $\mathcal{G}$.

Definition 3.2 (Observation-generated topology). The topology generated by Φ is

$$\tau_\Phi := \text{Top}\{\varsigma(\varphi_i) : i \in I\}$$

on the fixed carrier Ω_P . The displayed truth regions are a subbasis; their finite intersections form a basis. We call (Ω_P, τ_Φ) the *observation space generated by Φ* .

Its points are valuations and its designated subbasic opens are truth sets. The carrier is always displayed because a family of subsets is not a topology until its underlying set is fixed. The indexed family Φ and any assigned weights are presentation data; they are not part of the topology τ_Φ and need not be recoverable from it.

Because complete minterms define every subset of the finite carrier, every topology on Ω_P can be presented by choosing formulas for its open sets. Thus the propositional language supplies a finite presentation and interpretation; observation-generated spaces are not a new restricted class of finite topological spaces.

The empty intersection of subbasic opens is Ω_P . Hence a convenient basis is

$$\mathcal{B}_\Phi = \left\{ \bigcap_{i \in J} \varsigma(\varphi_i) : J \subseteq I \right\}.$$

Equivalently, give $\mathbb{S} = \{0, 1\}$ the Sierpiński topology $\{\emptyset, \{1\}, \mathbb{S}\}$. Then τ_Φ is the initial topology of the coordinate maps $b_i : \Omega_P \rightarrow \mathbb{S}$, or of their product $o_\Phi : \Omega_P \rightarrow \mathbb{S}^I$. This is the standard topological formulation of positive observable predicates [18, Example 1.12(3)].

Order the Boolean cube $\{0, 1\}^I$ coordinatewise, or equivalently order its subsets by inclusion.

Theorem 3.3 (Signature representation). *The opens of τ_Φ are precisely the inverse images of upward-closed sets of realized observation signatures:*

$$\tau_\Phi = \{o_\Phi^{-1}(U) : U \subseteq o_\Phi(\Omega_P) \text{ is upward closed}\}.$$

Proof. For a realized signature s , its principal upward set is

$$\uparrow s = \{t : s \subseteq t\}.$$

Its inverse image is the basic open

$$o_\Phi^{-1}(\uparrow s) = \bigcap_{i \in s} \varsigma(\varphi_i).$$

Every upward-closed set in a finite poset is the union of the principal upward sets of its members, so its inverse image is open. Conversely, a basic open indexed by any $J \subseteq I$ is the inverse image of the upward set

$$\{s \in o_\Phi(\Omega_P) : J \subseteq s\}.$$

Unions of basic opens give the reverse inclusion. $\square$

3.1 Partitions induced by observations

Define

$$v \sim_\Phi w \iff o_\Phi(v) = o_\Phi(w).$$

This equivalence relation partitions Ω_P into observational states.

Definition 3.4 (Partition topology). For a partition Π of a set S , define

$$\tau_\Pi := \left\{ \bigcup \mathcal{A} : \mathcal{A} \subseteq \Pi \right\}.$$

Its blocks are both open and closed.

An arbitrary observation topology need not be a partition topology. The difference is essential: partition topologies describe symmetric indistinguishability classes, whereas general finite topologies can also record one-way observational dependence.

Proposition 3.5 (Positive and two-sided observation). *Let $\Phi^\pm$ be indexed by $I \times \{+, -\}$, with $\varphi_{(i,+)} = \varphi_i$ and $\varphi_{(i,-)} = \neg\varphi_i$. Then $\tau_{\Phi^\pm}$ is the partition topology whose blocks are the $\sim_\Phi$ -classes.*

Proof. For every realized signature s , its fiber is

$$o_\Phi^{-1}(s) = \bigcap_{i \in s} \varsigma(\varphi_i) \cap \bigcap_{i \in I \setminus s} \varsigma(\neg\varphi_i).$$

It is therefore open in $\tau_{\Phi^\pm}$. Every open generated by $\Phi^\pm$ is saturated under equality of signatures, so it is a union of these fibers. $\square$

Example 3.6 (Two positive observations). Take $\Phi = (X, Y)$ on $\Omega_{X,Y}$. Then

$$\tau_\Phi = \{\emptyset, \{B\}, \{A, B\}, \{B, C\}, \{A, B, C\}, \Omega_{X,Y}\}.$$

Adding $\neg X$ and $\neg Y$ makes each of A, B, C, D open and gives the discrete topology.

For later reference, Table 2 collects the complete signatures and smallest positive-observation neighborhoods for this running example. The last two columns anticipate the implication and XOR examples without changing their classical truth values.

Cell	$o_{(X,Y)}$	$N_{(X,Y)}$	$X \Rightarrow Y$	$X \Updownarrow Y$
A	$(1, 0)$	$\{A, B\}$	F	T
B	$(1, 1)$	$\{B\}$	T	F
C	$(0, 1)$	$\{B, C\}$	T	T
D	$(0, 0)$	$\Omega_{X,Y}$	T	F

Table 2: The four-cell example used throughout the paper.

4 Distinguishability and separation

Three questions should not be conflated:

1. Formulas $\mathcal{L}_1, \mathcal{L}_2$ are semantically different when $\varsigma(\mathcal{L}_1) \neq \varsigma(\mathcal{L}_2)$.
2. Valuations v, w are observationally distinguishable by Φ when $o_\Phi(v) \neq o_\Phi(w)$.
3. Points are topologically separated according to the neighborhoods available in τ_Φ .

Overlap between two truth regions does not by itself establish topological indistinguishability.

Proposition 4.1 (Semantic XOR criterion). *For formulas $\mathcal{L}_1, \mathcal{L}_2$,*

$$\varsigma(\mathcal{L}_1) \neq \varsigma(\mathcal{L}_2) \iff \varsigma(\mathcal{L}_1 \Updownarrow \mathcal{L}_2) \neq \emptyset.$$

At a particular valuation v , their truth values are distinct exactly when $v \in \varsigma(\mathcal{L}_1 \Updownarrow \mathcal{L}_2)$.

Proof. The XOR truth region is the symmetric difference

$$\varsigma(\mathcal{L}_1) \triangle \varsigma(\mathcal{L}_2).$$

It is nonempty exactly when the two truth regions are unequal. □

Observing only $\mathcal{L}_1 \wedge \mathcal{L}_2$ gives one positive bit and groups together all three ways in which the conjunction can be false. Replacing that observation by the pair $\Phi = (\mathcal{L}_1, \mathcal{L}_2)$ refines those states according to the realized truth vectors $(1, 1), (1, 0), (0, 1), (0, 0)$. Adding both negations makes the nonempty truth-pattern cells clopen. This is the precise observational sense in which the conjuncts become distinguishable.

Definition 4.2 (Topological indistinguishability). Points $v, w \in \Omega_P$ are topologically indistinguishable when they belong to exactly the same open sets of τ_Φ .

Theorem 4.3 (Indistinguishability theorem). *For $v, w \in \Omega_P$, the following are equivalent:*

- (i) v and w are topologically indistinguishable in τ_Φ ;
- (ii) v and w have the same subbasic open memberships;
- (iii) $o_\Phi(v) = o_\Phi(w)$;
- (iv) $v \sim_\Phi w$.

Proof. Equality of all open memberships implies equality of subbasic memberships. Subbasic memberships are exactly the coordinates of o_Φ . Conversely, equality of signatures gives equality of membership in every finite intersection of subbasic opens and hence in every union of such intersections. □

Corollary 4.4 (T_0 quotient). *The space (Ω_P, τ_Φ) is T_0 exactly when o_Φ is injective. Its Kolmogorov quotient is naturally identified with the set $o_\Phi(\Omega_P)$ of realized signatures, equipped with the subspace upper-set topology inherited from $\mathbb{S}^I$.*

The next separation level requires distinguishability in both directions.

Proposition 4.5 (T_1 and complete finite separation). *The observation space is T_1 exactly when, for every distinct v, w ,*

$$o_\Phi(v) \not\subseteq o_\Phi(w) \quad \text{and} \quad o_\Phi(w) \not\subseteq o_\Phi(v).$$

Equivalently, o_Φ is injective and its image is an antichain. A finite T_1 space is discrete; therefore finite Hausdorff observation spaces are discrete as well.

Proof. There is an open containing v but not w exactly when some observation is true at v and false at w , which is equivalent to $o_\Phi(v) \not\subseteq o_\Phi(w)$. The T_1 condition requires this in both directions. In a finite T_1 space, each singleton is the finite intersection of open complements of the other singletons, so every singleton is open. $\square$

Example 4.6 (Sierpiński observation space). Let $P = \{X\}$ and $\Phi = (X)$. With valuations 0, 1, the topology is

$$\tau_\Phi = \{\emptyset, \{1\}, \{0, 1\}\}.$$

It is T_0 but not T_1 . The positive observation X distinguishes 1 from 0, but no positive observation in Φ distinguishes 0 from 1. This is the canonical two-point example of asymmetric observation.

5 Asymmetric separation and the specialization preorder

We use the specialization convention

$$v \preceq_\Phi w \iff \text{every open neighborhood of } v \text{ contains } w.$$

Some authors use the reverse convention; the direction is fixed here by the display above.

Because Ω_P is finite, every point has a smallest open neighborhood

$$N_\Phi(v) := \bigcap \{U \in \tau_\Phi : v \in U\}.$$

Theorem 5.1 (Positive-information order). *For all $v, w \in \Omega_P$,*

$$v \preceq_\Phi w \iff o_\Phi(v) \subseteq o_\Phi(w).$$

Moreover,

$$N_\Phi(v) = \bigcap_{i \in o_\Phi(v)} \varsigma(\varphi_i) = \{w : o_\Phi(v) \subseteq o_\Phi(w)\}.$$

Proof. If every open containing v contains w , this holds for every subbasic open, so every observation true at v is true at w . Conversely, if $o_\Phi(v) \subseteq o_\Phi(w)$, then every basic open containing v contains w , and therefore every open containing v contains w . The neighborhood formula follows. $\square$

Thus w preserves all positive information observable at v and may satisfy additional observations. Failure of symmetry is not a defect: it records that positive evidence can separate one direction without separating the reverse. After quotienting by $\sim_\Phi$, the preorder becomes the inclusion order on realized signatures.

6 Distances on observation signatures

Coordinatewise XOR naturally produces a difference vector. Let $\mathbf{a} = (a_i)_{i \in I}$ be positive real weights. As noted in Section 3, these formulas use complete semantic profiles, even when the topology is read as an interface for affirmative certificates. The distances are deliberately presentation-dependent: changing weights, duplicating coordinates, or replacing Φ by a different family that generates the same topology can change their numerical values.

Definition 6.1 (Observable XOR difference). Define

$$\Delta_\Phi(v, w) := o_\Phi(v) \updownarrow o_\Phi(w) := (b_i(v) \updownarrow b_i(w))_{i \in I}.$$

This vector identifies exactly which observations disagree.

Definition 6.2 (Symmetric signature distance). Define the weighted Hamming distance between signatures by

$$d_\Phi(v, w) := \sum_{i \in I} a_i |b_i(v) - b_i(w)|.$$

This is a standard weighted form of Hamming distance [9].

Proposition 6.3. *The function d_Φ is a pseudometric on Ω_P , and*

$$d_\Phi(v, w) = 0 \iff v \sim_\Phi w.$$

Consequently, d_Φ induces a metric on the underlying set $\Omega_P / \sim_\Phi$. If τ_{d_Φ} denotes the pseudometric topology on Ω_P , then

$$\tau_{d_\Phi} = \tau_{\Phi^\pm}.$$

The induced metric topology on the finite quotient set is discrete and need not equal its Kolmogorov quotient topology.

Proof. Each coordinate contributes the weighted discrete metric on $\{0, 1\}$; their sum is a pseudometric. It vanishes exactly when every observation coordinate agrees. If $I \neq \emptyset$, every sufficiently small ball at v is exactly the signature fiber $[v]_\Phi$, while every ball is a union of such fibers. Hence the pseudometric topology is the partition topology of Proposition 3.5. For $I = \emptyset$ there is one fiber and both topologies are indiscrete. A genuine metric on a finite set has discrete topology, giving the final qualification. $\square$

For example, the Sierpiński space of Example 4.6 is already T_0 , so its Kolmogorov quotient has the same nondiscrete topology. Its Hamming metric is the ordinary discrete two-point metric. Thus “metric on the quotient set” does not mean “metric inducing the quotient topology.”

Symmetric distance measures disagreement but cannot by itself encode the direction of the specialization preorder. For that purpose define the cost of losing observations.

Definition 6.4 (Directed loss distance). Define

$$d_\Phi^+(v, w) := \sum_{i \in I} a_i \mathbf{1}[b_i(v) = 1 \text{ and } b_i(w) = 0].$$

Equivalently, this is weighted set difference from the support of v to the support of w . In XOR notation the direction is supplied by the source mask:

$$d_\Phi^+(v, w) = \sum_{i \in I} a_i b_i(v) (b_i(v) \updownarrow b_i(w)).$$

This finite weighted sum is an elementary directed set-difference construction. Pavlovic uses the cardinality $|y \setminus x|$, with the opposite argument orientation from our loss convention [13, Sec. 2], while de Brecht uses a supremum of decaying weights on a countable powerset [5, Sec. 4].

The theorem below is therefore an explicit coordinate representation for the chosen observation presentation, not a claim that quasi-metrizability of finite Alexandrov spaces is new. In particular, Dovgoshey and Shanin prove that every T_0 Alexandrov space is quasi-metrizable by an equidistant quasi-metric [7]. Here d_Φ^+ works already before the T_0 quotient, at the price of being a quasi-pseudometric when distinct points have the same signature.

Theorem 6.5 (Quasi-pseudometric representation). *The function d_Φ^+ is a quasi-pseudometric: it is nonnegative, $d_\Phi^+(v, v) = 0$, and*

$$d_\Phi^+(v, u) \leq d_\Phi^+(v, w) + d_\Phi^+(w, u).$$

It also satisfies

$$d_\Phi^+(v, w) = 0 \iff v \preceq_\Phi w,$$

and

$$d_\Phi(v, w) = d_\Phi^+(v, w) + d_\Phi^+(w, v).$$

The topology generated by the forward balls, for $r > 0$,

$$B^+(v, r) := \{w : d_\Phi^+(v, w) < r\}$$

is exactly τ_Φ .

Proof. For each coordinate, an observation lost from v to u must be lost either from v to w or from w to u . Summing the resulting coordinatewise inequality proves the triangle inequality. Vanishing means that no observation true at v is false at w , which is exactly $o_\Phi(v) \subseteq o_\Phi(w)$. The symmetrization identity follows because every disagreement is a loss in exactly one direction.

Every forward ball is upward closed in $\preceq_\Phi$: if w belongs to the ball and $w \preceq_\Phi u$, then u loses no more observations from v than w does. Hence every forward ball is τ_Φ -open. If $I \neq \emptyset$ and $0 < r \leq \min_{i \in I} a_i$, then

$$B^+(v, r) = N_\Phi(v).$$

The smallest neighborhoods generate τ_Φ , so the forward-ball topology is exactly τ_Φ . If $I = \emptyset$, the directed distance is identically zero and every positive-radius forward ball is Ω_P , again giving the correct indiscrete topology. $\square$

The zero-distance relation of d_Φ^+ recovers the preorder and its forward balls recover the topology. Its positive numerical values contain additional presentation data: changing weights, or adding a topologically redundant coordinate, can change distances without changing $\preceq_\Phi$ or τ_Φ . This is compatible with the general role of nonsymmetric distance in generalized metric theory [10].

7 Interior, closure, and observational boundary

Topological operators describe what can be verified or ruled out using the chosen observations.

Definition 7.1. For an event $E \subseteq \Omega_P$, let

$$\text{Int}_\Phi(E) := \text{Int}_{\tau_\Phi}(E), \quad \text{Cl}_\Phi(E) := \text{Cl}_{\tau_\Phi}(E), \quad \partial_\Phi E := \text{Cl}_\Phi(E) \setminus \text{Int}_\Phi(E).$$

The interior $\text{Int}_\Phi(E)$ is the largest observable open event contained in E . If the actual valuation lies in this interior, a finite conjunction of available positive observations certifies membership in E . The closure $\text{Cl}_\Phi(E)$ consists of valuations at which the available observations cannot exclude contact with E . The boundary $\partial_\Phi E$ is therefore an observational ambiguity region. Equivalently,

$$\partial_\Phi E = \text{Cl}_\Phi(E) \cap \text{Cl}_\Phi(\Omega_P \setminus E).$$

It contains precisely those points whose available neighborhoods cannot settle membership in E one way or the other.

Proposition 7.2 (Finite neighborhood formulas). *For every $E \subseteq \Omega_P$,*

$$\begin{aligned} \text{Int}_\Phi(E) &= \{v : N_\Phi(v) \subseteq E\}, \\ \text{Cl}_\Phi(E) &= \{v : N_\Phi(v) \cap E \neq \emptyset\}. \end{aligned}$$

Consequently,

$$v \in \partial_\Phi E \iff N_\Phi(v) \cap E \neq \emptyset \text{ and } N_\Phi(v) \cap (\Omega_P \setminus E) \neq \emptyset.$$

Proof. The point v is interior to E exactly when it has an open neighborhood contained in E . Since $N_\Phi(v)$ is its smallest neighborhood, this is equivalent to $N_\Phi(v) \subseteq E$. A point belongs to the closure of E exactly when each of its neighborhoods meets E , which in a finite space is equivalent to $N_\Phi(v) \cap E \neq \emptyset$. $\square$

Remark 7.3 (Boundary versus XOR). The symbol ∂_Φ denotes the topological boundary operator above. It is not the Boolean symmetric-difference operation that removes the overlap of two formulas. That Boolean operation is the exclusive remainder

$$\varsigma(\mathcal{L}_1 \vee \mathcal{L}_2) \setminus \varsigma(\mathcal{L}_1 \wedge \mathcal{L}_2) = \varsigma(\mathcal{L}_1 \uplus \mathcal{L}_2).$$

The two constructions can differ even in the simplest cases: in a discrete topology every topological boundary is empty, while an XOR truth region may be nonempty.

Example 7.4 (Boundary of implication under positive observations). For $\Phi = (X, Y)$,

$$E = \varsigma(X \Rightarrow Y) = \{B, C, D\}.$$

Using the topology in Example 3.6,

$$\text{Int}_\Phi(E) = \{B, C\}, \quad \text{Cl}_\Phi(E) = \Omega_{X,Y}, \quad \partial_\Phi E = \{A, D\}.$$

Thus $X \Rightarrow Y$ is neither open nor closed in this observation topology. The point D satisfies the implication but is not positively certifiable as such using only X and Y , while A lies observationally adjacent to satisfying states despite falsifying the implication.

Example 7.5 (XOR region under coarse and discrete observation). Retain the partition labels $A = X \wedge \neg Y$, $B = X \wedge Y$, $C = \neg X \wedge Y$, and $D = \neg X \wedge \neg Y$. The exclusive truth region is

$$E_\oplus := \varsigma(X \uplus Y) = \{A, C\}.$$

First use only the positive observations $\Phi = (X, Y)$. The smallest neighborhoods are

$$N_\Phi(A) = \{A, B\}, \quad N_\Phi(B) = \{B\}, \quad N_\Phi(C) = \{B, C\}, \quad N_\Phi(D) = \Omega_{X,Y}.$$

Hence no nonempty τ_Φ -open set is contained in $E_\oplus$, while the neighborhoods of A , C , and D each meet it. Thus

$$\text{Int}_\Phi(E_\oplus) = \emptyset, \quad \text{Cl}_\Phi(E_\oplus) = \{A, C, D\}, \quad \partial_\Phi E_\oplus = \{A, C, D\}.$$

The points A and C belong to the XOR region but are not positively certifiable using only X and Y , because their smallest neighborhoods also contain B . The point D is not in the XOR region, but it remains on its boundary because every positive-observation neighborhood of D contains both XOR and non-XOR valuations.

Now take the two-sided observation family

$$\Phi^\pm = (X, \neg X, Y, \neg Y).$$

Its topology is discrete. Therefore $E_\oplus$ is both open and closed, and

$$\text{Int}_{\Phi^\pm}(E_\oplus) = \text{Cl}_{\Phi^\pm}(E_\oplus) = \{A, C\}, \quad \partial_{\Phi^\pm} E_\oplus = \emptyset.$$

The XOR truth set is unchanged by the topology; only its observable boundary changes. This is the distinction between Boolean symmetric difference and topological ambiguity. Figure 1 displays the same calculation on the four truth-pattern cells.

	X	$\neg X$
Y	$B = X \wedge Y$ non-XOR	$C = \neg X \wedge Y$ XOR
$\neg Y$	$A = X \wedge \neg Y$ XOR	$D = \neg X \wedge \neg Y$ non-XOR

Figure 1: The four truth-pattern cells for X and Y . The bold cells form $E_\oplus = \varsigma(X \uparrow Y) = \{A, C\}$. Under positive observation $\Phi = (X, Y)$, the observable boundary is $\partial_\Phi E_\oplus = \{A, C, D\}$; under $\Phi^\pm$ it is empty.

7.1 Open truth regions and Heyting implication

The opens of any topology form a Heyting algebra. For $U, V \in \tau_\Phi$, define

$$U \Rightarrow_H V := \text{Int}_\Phi((\Omega_P \setminus U) \cup V), \quad \neg_H U := \text{Int}_\Phi(\Omega_P \setminus U).$$

Then

$$U \cap (U \Rightarrow_H V) \subseteq V$$

is topological Modus Ponens, and

$$(U \Rightarrow_H V) \cap \neg_H V \subseteq \neg_H U$$

is topological Modus Tollens. In a discrete topology the interiors do nothing, and these reduce to their classical truth-set forms. In a nondiscrete observation topology, the interior records the loss imposed by requiring the result to remain observable.

Indeed, for every open W ,

$$W \subseteq U \Rightarrow_H V \iff W \cap U \subseteq V,$$

which is the defining Heyting adjunction. The two displayed inference inequalities follow from it. These open-set operations belong to standard topological semantics [11, 18]; they should be distinguished from the classical truth-region operations used earlier. A full intuitionistic interpretation would assign open sets to atoms and evaluate compound formulas recursively in this Heyting algebra.

For example, in the Sierpiński space of Example 4.6, interpret X by the open $\{1\}$. Then $\neg_H\{1\} = \emptyset$ and $\neg_H\neg_H\{1\} = \Omega_X$, whereas the interior of the classical truth region $\varsigma(\neg\neg X) = \{1\}$ is still $\{1\}$. Recursive Heyting evaluation therefore differs from taking one final interior of a classical truth region.

8 Information refinement and logical update

The phrase “acquiring information” can describe two mathematically different operations.

8.1 Refining the observations

Let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$, where $I \subseteq J$ and $\psi_i = \varphi_i$ for every $i \in I$. Thus Ψ extends Φ while retaining the identities of its old coordinates.

Theorem 8.1 (Refinement theorem). *For such an indexed extension $\Phi \hookrightarrow \Psi$:*

- (i) $\tau_\Phi \subseteq \tau_\Psi$;
- (ii) $v \sim_\Psi w$ implies $v \sim_\Phi w$;
- (iii) $\preceq_\Psi \subseteq \preceq_\Phi$ as relations;
- (iv) if Ψ retains the weights a_i on $i \in I$, then $d_\Psi^+(v, w) \geq d_\Phi^+(v, w)$ and $d_\Psi(v, w) \geq d_\Phi(v, w)$.

Proof. The subbasis for τ_Φ is contained in the subbasis for τ_Ψ , proving (i). Equality of the longer signatures implies equality after deleting the new coordinates, proving (ii). Inclusion of Ψ -signatures implies inclusion after deleting those coordinates, proving (iii). The added coordinates make nonnegative contributions to both distances, proving (iv). $\square$

Adding an observable can split an indistinguishability block, remove a one-way comparison, and increase measured separation. A proposed observation ψ is topologically redundant relative to Φ exactly when

$$\varsigma(\psi) \in \tau_\Phi,$$

because adding an already open truth region does not refine the topology. Topological redundancy need not mean numerical redundancy: an added coordinate can add a further term to d_Φ and d_Φ^+ while leaving the generated topology unchanged. For example, adding $X \wedge Y$ to $\Phi = (X, Y)$ is topologically redundant because $\varsigma(X \wedge Y)$ is already open. With unit weights it nevertheless changes $d^+(B, A)$ from 1 to 2.

8.2 Updating the possible valuations

Let $S \subseteq \Omega_P$ be the current possibility region. Learning a premise γ produces

$$S \longmapsto S \cap \varsigma(\gamma).$$

For premises $\Gamma = \{\gamma_1, \dots, \gamma_r\}$, define

$$S[\Gamma] := S \cap \bigcap_{j=1}^r \varsigma(\gamma_j), \quad S_\Gamma := \Omega_P[\Gamma].$$

For an arbitrary current carrier S , the updated observation space is

$$\mathbf{X}_{S,\Gamma}^T := (S[\Gamma], \tau_\Phi|_{S[\Gamma]}), \quad \tau_\Phi|_S := \{U \cap S : U \in \tau_\Phi\}.$$

When $S = \Omega_P$, write $\mathbf{X}_\Gamma^T := \mathbf{X}_{\Omega_P,\Gamma}^T$. Sequential updates are associative, commutative, and idempotent because set intersection has those properties. Inconsistent premises produce the empty space. Smallest neighborhoods restrict accordingly:

$$N_{\Phi|S[\Gamma]}(v) := S[\Gamma] \cap N_\Phi(v) \quad (v \in S[\Gamma]).$$

Observation refinement and premise update need not have the same effect. Refinement changes which distinctions are expressible; update changes which worlds remain possible.

9 Inference by truth-region restriction

Classical inference is a relationship between the common truth region of the premises and the truth region of the conclusion.

Theorem 9.1 (Semantic consequence). *For a finite set of premises Γ and a conclusion ψ ,*

$$\Gamma \models \psi \iff S_\Gamma \subseteq \varsigma(\psi).$$

When this holds, the inclusion

$$\mathbf{X}_\Gamma^T \hookrightarrow (\varsigma(\psi), \tau_\Phi|_{\varsigma(\psi)})$$

is continuous.

Proof. The semantic statement says exactly that every valuation satisfying all premises satisfies ψ . The inclusion between two subspaces of the same ambient space is continuous. $\square$

The topology contributes a structured representation of this consequence: the common premise region carries an induced observation topology, and the valid inclusion is a continuous map into the conclusion subspace. It does not alter which consequences are classically valid. In particular, premise conjunction uses intersection of truth regions (or the pullback of their induced subspaces), not intersection of two families of open sets. The latter is a different operation: a meet in the lattice of topologies on one fixed carrier.

9.1 A remark on minimal truth spaces

For a truth region S , one may form the indiscrete space $\mathcal{T}_S^T = (S, \{\emptyset, S\})$. Intersecting carriers and then assigning this indiscrete topology transports the ordinary meet $R \cap S$ of the power set into new notation; it supplies no additional topological inference principle. Moreover, $\mathcal{T}_S^T$ generally differs from the induced observation subspace $(S, \tau_\Phi|_S)$, and its inclusion into the ambient observation space need not be continuous. It therefore cannot replace the induced subspaces in the pullback formulation below. The two spaces coincide exactly when the induced topology on S is indiscrete, as for a singleton or a region whose points all have the same observation signature.

9.2 Modus Ponens and Modus Tollens

Using A, B, C, D ,

$$\begin{aligned}\varsigma(X \Rightarrow Y) \cap \varsigma(X) &= \{B\} = \varsigma(X \wedge Y) \subseteq \varsigma(Y), \\ \varsigma(X \Rightarrow Y) \cap \varsigma(\neg Y) &= \{D\} = \varsigma(\neg X \wedge \neg Y) \subseteq \varsigma(\neg X).\end{aligned}$$

9.3 Hypothetical Syllogism

Write a three-variable valuation as the triple (X, Y, Z) . Then

$$\varsigma(X \Rightarrow Y) \cap \varsigma(Y \Rightarrow Z) = \{(0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 1, 1)\} \subsetneq \varsigma(X \Rightarrow Z).$$

The inclusion is proper: $(1, 0, 1)$ satisfies $X \Rightarrow Z$ but not $X \Rightarrow Y$. The two premises therefore carry more information than the conclusion; validity requires inclusion, not equality.

9.4 Pullback formulation

For a single formula $\mathcal{L}$, write

$$\mathbf{X}_{\mathcal{L}}^T := (\varsigma(\mathcal{L}), \tau_{\Phi}|_{\varsigma(\mathcal{L})}).$$

The inclusions of two truth subspaces into (Ω_P, τ_{Φ}) have pullback

$$\mathbf{X}_{\mathcal{L}_1}^T \times_{(\Omega_P, \tau_{\Phi})} \mathbf{X}_{\mathcal{L}_2}^T \cong \mathbf{X}_{\mathcal{L}_1 \wedge \mathcal{L}_2}^T.$$

For Modus Ponens,

$$\mathbf{X}_{X \Rightarrow Y}^T \times_{(\Omega_{X, Y}, \tau_{\Phi})} \mathbf{X}_X^T \cong \mathbf{X}_{X \wedge Y}^T \hookrightarrow \mathbf{X}_Y^T.$$

Because both arrows are subspace inclusions, this pullback is just the set-theoretic intersection equipped with its subspace topology, via the diagonal identification $v \mapsto (v, v)$. Its universal property is the standard one for intersections of subspaces: compatible maps into the two premise regions factor uniquely and continuously through their common region. Thus the categorical notation repackages premise conjunction; it does not add a new inference rule.

10 Continuous maps and preservation of information

Let $\Phi = (\varphi_i)_{i \in I}$ and $\Psi = (\psi_j)_{j \in J}$ generate the observation spaces (Ω_P, τ_{Φ}) and (Ω_Q, τ_{Ψ}) . A map

$$f : \Omega_P \longrightarrow \Omega_Q$$

is continuous when the inverse image of every Ψ -observable open event is Φ -observable. It suffices to check the subbasic truth regions:

$$f^{-1}(\varsigma(\psi_j)) \in \tau_{\Phi} \quad (j \in J).$$

Proposition 10.1 (Monotonicity of continuous maps). *If f is continuous and $v \preceq_{\Phi} w$, then*

$$f(v) \preceq_{\Psi} f(w).$$

Proof. Let $U \in \tau_{\Psi}$ contain $f(v)$. Then $f^{-1}(U)$ is a Φ -open set containing v , so it contains w . Hence U contains $f(w)$. $\square$

Continuity therefore preserves one-way observational dependence. This gives a rigorous starting point for continuous valuation maps. Conversely, because these finite spaces have their upper-set topologies, every monotone valuation map is continuous. Continuity may collapse distinguishable points and does not by itself imply preservation of either weighted distance.

11 Scope, limitations, and deferred directions

The results above form a finite expository theory of observation, distinguishability, distance, refinement, and inference. They do not establish new homological or application-specific results. The most direct extensions are questions about the design and use of observations in reasoning.

11.1 A presentation-dependent nerve

A standard relation-nerve construction can be applied to a family of truth regions. For an indexed observation family $\Phi = (\varphi_i)_{i \in I}$, define a simplicial complex

$$K_\Phi := \left\{ J \subseteq I : \bigcap_{i \in J} \varsigma(\varphi_i) \neq \emptyset \right\}.$$

Its vertices are the indices of individually satisfiable observations, and its simplices are jointly satisfiable indexed subfamilies. This is the observation-side Dowker complex of the satisfaction relation [6]. It can record higher-order compatibility not visible from pairwise intersections, but it is not an invariant of τ_Φ . For example, $\Phi = (X, Y, \neg(X \wedge Y))$ gives the boundary of a triangle. Its observation regions are $\{A, B\}$, $\{B, C\}$, and $\{A, C, D\}$: their pairwise intersections are respectively $\{B\}$, $\{A\}$, and $\{C\}$, while their triple intersection is empty. Thus the nerve is connected, whereas the observation space has connected components $\{B\}$ and $\{A, C, D\}$. Adjoining a tautology leaves the generated topology and both signature distances unchanged, but makes the nerve a cone and kills its reduced homology. Any homological use must therefore state the presentation and the intended invariance separately: the homology of K_Φ need not agree with the singular homology of the observation space and is not determined by τ_Φ .

11.2 Focused directions beyond the synthesis

The most natural next step is an observation-design problem rather than a new name for the finite topology itself: find a smallest or least-cost observation family that separates designated valuations, certifies a target conclusion, or distinguishes competing premise sets. Such questions could support genuinely new algorithmic or complexity results. A second direction is to study sequences in which observation refinement and premise restriction alternate, or to compare the classical truth-region semantics used here with the Heyting semantics of the open-set algebra. These are deliberately left outside the present expository scope.

12 Conclusion

An observation-generated topology can be nontrivial when it records a selected family of formulas rather than every definable truth set. Observation signatures simultaneously determine a partition into indistinguishable states, a finite topology, and its specialization preorder. Coordinatewise XOR records which observations differ; weighted Hamming distance measures symmetric disagreement; and the directed distance d_Φ^+ measures loss of positive information while generating the selected observation topology. Adding tests refines distinguishability, while adding premises restricts the possibility space. Inference is the inclusion of that restricted region in the truth region of a conclusion.

The framework gives a common notation for partitioning, distinguishability, asymmetric separation, logical distance, and refinement. Its individual ingredients are classical. Its value here is expository: it makes explicit which structures are determined by the observations, which require

extra weights or presentation choices, and which logical claims remain ordinary model-set inclusions. Natural research extensions would add an algorithmic or complexity result, a new invariant, or a validated application rather than more renamings of the same finite structures.

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