Terminal response and worst-case bounded-run switching near a parabolic critical branch

7 November 2026

Abstract

We study planar monotone switching maps in which one mode omits a quadratic contraction, the modes share an increasing second-coordinate update, and both preserve the first coordinate after threshold crossing. From an exact baseline critical state with threshold deficit ρ, we prove the schedule-uniform terminal estimate $|R - S| \le C_0\rho^7 + C_0\rho^2 S$, where S is the sum of omitted contractions before crossing. It yields a delayed-intervention margin asymptotic $M_n \sim Cn^{-5/2}$. An elementary interchange inequality identifies the exact minimum-margin schedule under a bound of $K-1$ consecutive baseline updates. A complete-tail estimate then gives $G_K = P_K \sim C\zeta(5/2)K^{-5/2}$. The result allows arbitrary higher coefficients in two finite-regularity functions while retaining quadratic flatness and terminal freezing. The prototype's critical geometry has established antecedents; the contribution concerns terminal transfer and constrained switching. A leakage example shows why leading weighted local terms alone cannot determine the terminal observable.

1 Introduction and relation to prior work

A small intervention near a slowly attracting critical orbit may leave a terminal effect that is much smaller than the state itself. Local orbit asymptotics suggest the size of an intervention, but do not by themselves identify its eventual effect after a long excursion. Repeated interventions pose a second difficulty: convergence on each fixed collection of blocks does not justify an asymptotic formula for the whole infinite schedule. We address these issues for an explicit class in which an intervention omits contraction in one coordinate and leaves the other update unchanged.

The local critical geometry in the simplest member of this class has a precise antecedent. For the pre-crossing map $r' = r(1 - ad^2)$, $d' = d - br$, the change $x = d$, $y = -br$ gives $(x, y) \mapsto (x + y, y - ax^2y)$. This is a member of the analytic normal-form family in Jamieson and Merino [1]. Their Theorem A, recalling a result of Casasayas, Fontich and Nunes, gives the cubic invariant-curve coefficient; their Theorem 3 studies the surrounding sectors. This comparison concerns the analytic pre-crossing map, not its frozen continuation. Cufí-Cabré and Fontich [2] develop invariant-curve approximation and finite-regularity theory for nilpotent parabolic points. In particular, high endpoint regularity of an invariant curve should not be presumed. Our proof uses elementary order arguments and a polynomial approximate invariant, without differentiating a critical graph.

Comparison of ordered dynamics belongs to the general theory of monotone systems; see Angeli and Sontag [3] for a continuous-time control framework. The finite-word interchange used here is proved directly. The accumulation of local perturbative effects also has a broad precedent: canonical discrete Melnikov theory [4] uses orbit sums to describe manifold displacement in a normally hyperbolic setting. It does not directly give the parabolic terminal estimate proved below. We do not claim new general methods for invariant manifolds, monotone control, or summing perturbations. Nor is the appearance of a zeta value itself the contribution.

Our main estimate controls terminal transfer uniformly in the schedule and its possibly long crossing time. It implies a one-intervention law. In combination with a legal-word interchange and

a summable bound for the entire remaining intervention tail, it also gives an exact bounded-run optimizer and its asymptotic margin. These statements concern the declared structured class. They are not a universality theorem for arbitrary monotone maps or perturbations that change the order of flatness.

2 Model, observables, and results

Fix $0 < \theta < 1$. Initially let $f \in C^6([0, \delta_0])$ and $g \in C^4([0, \delta_0])$, with one-sided derivatives at zero, satisfy

$$f(d) = ad^2 + O(d^3), \qquad g(d) = b + O(d), \qquad a, b > 0. \tag{1}$$

Extend g to a positive Lipschitz function $\bar{g}$ on $[\theta - 1, \delta_0]$, agreeing with g for $d \geq 0$. Put $F(d) = f(d)$ for $d \geq 0$ and $F(d) = 0$ for $d < 0$. Choose $r_* > 0$ and then $\delta > 0$ sufficiently small, as specified in Lemma 3. On

$$\mathcal{R} = (0, r_*] \times [\theta - \delta, 1]$$

define

$$\begin{aligned} W(r, s) &= \min\{1, s + \bar{g}(\theta - s)r\}, \\ B(r, s) &= (r[1 - F(\theta - s)], W(r, s)), \qquad A(r, s) = (r, W(r, s)). \end{aligned} \tag{2}$$

Here B is the baseline mode and A is an intervention. We choose $r_* < \theta - \delta$, $L_g r_* < 1$, and $0 < f(d) < 1$, $f'(d) > 0$ for $0 < d \leq \delta$, where L_g is a Lipschitz constant of $\bar{g}$. Both maps preserve $\mathcal{R}$ and are isotone in the coordinatewise order on (r, s). Indeed, F is nondecreasing, and the uncapped second update has increments in s bounded below by $(1 - L_g r_*)$ times the increment in s; it is also increasing in r. Taking a minimum with 1 preserves isotonicity. The first coordinate never increases and the second never decreases. For $s \geq \theta$, both modes freeze r.

We write $d = \theta - s$. At prestates with $d > 0$, the uncapped updates are

$$B : (r, d) \mapsto (r[1 - f(d)], d - g(d)r), \qquad A : (r, d) \mapsto (r, d - g(d)r). \tag{3}$$

The order used throughout is that on (r, s), not on (r, d). A critical state is a state whose all-B orbit has $r_t \to 0$ and $d_t \downarrow 0$ without reaching $d = 0$ at any finite time. Lemma 3 proves existence and uniqueness on each small positive-d slice.

A schedule is an infinite word $\omega \in \{A, B\}^{\mathbb{N}_0}$. Its symbol ω_t updates state z_t to z_{t+1}. Fix a critical initial state z_0. Define its full-trajectory margin by

$$m(\omega; z_0) = \tfrac{1}{2} \inf_{t \geq 0} \min\{r_t, s_t\}.$$

This is an observation functional of an exact trajectory, with normalization $1/2$; no disturbance is fed back into the recurrence. Since $r_0 < s_0$, write $R = \lim_{t \to \infty} r_t$ and obtain $m(\omega; z_0) = R/2$. If τ is the first threshold hit, terminal freezing gives $R = r_\tau$; otherwise R still denotes the monotone limit. Define

$$M_n = \inf_{\eta \in \{A,B\}^{\mathbb{N}_0}} m(B^n A \eta; z_0), \qquad p_K = (B^{K-1} A)^\infty, \quad P_K = m(p_K; z_0),$$

and, for integers $K \geq 2$,

$$G_K = \inf\{m(\omega; z_0) : \omega \text{ has no } K \text{ consecutive } B\text{'s, including initially}\}.$$

All prefixes, including time zero, are part of these observables.

Theorem 1 (Uniform terminal transfer)**.** *For the local choices above, let a trajectory start at any critical state $(r, \theta - \rho)$, $0 < \rho \le \delta$, and use a schedule containing at least one A. It crosses $s \ge \theta$ at a finite time τ and has a positive terminal first coordinate $R = r_\tau$. Set*

$$S = \sum_{\substack{0 \le t < \tau \\ \omega_t = A}} r_t f(d_t).$$

There is a constant C_0, depending only on the fixed model and local rectangle, such that, uniformly in ρ, the schedule, and τ,

$$|R - S| \le C_0 \rho^7 + C_0 \rho^2 S. \tag{4}$$

In particular $R = S + O(\rho^7)$ for any family with $S = O(\rho^5)$. The latter condition is not asserted for arbitrary schedules.

Theorem 2 (Delayed intervention and bounded runs)**.** *Fix any exact critical initial state in $\mathcal{R}$. Put*

$$D = \sqrt{\frac{3}{2a}}, \qquad C = \frac{a^2 D^5}{6b}.$$

Then B^∞ attains the tail infimum defining M_n, and

$$M_n \sim C n^{-5/2}, \qquad G_K = P_K \sim C\zeta(5/2) K^{-5/2}. \tag{5}$$

The equality $G_K = P_K$ holds for every integer $K \ge 2$. The asymptotics fix the model, the continuation, and the exact initial state; they do not assert uniformity under varying parameters or initial states.

3 Critical states and their rates

Lemma 3 (Local critical branch)**.** *The choices of r_*, δ can be made so that, for every $0 < d \le \delta$, there is a unique critical first coordinate $q(d) > 0$. States on that slice with $r > q(d)$ cross under B, while those with $r < q(d)$ are noncrossing and have a positive limiting deficit. For some fixed L_0, $q(d) \le L_0 d^3$.*

Proof. First fix positive bounds $g_- \le \bar{g} \le g_+$ on a reduced compact interval, and $f(d) \le q_+ d^2$. Set $L_0 = q_+/g_-$, choose $M > L_0$ and $0 < k < a/(3b)$. Choose $r_* < \theta/2$ with $L_g r_* < 1$. Then reduce $\delta \le 1$ so that $\theta - \delta > r_*$, $M\delta^3 < r_*$, the stated conditions on f hold, and

$$g_+ k \delta^2 < 1, \qquad g_+ L_0 \delta^2 \le 1, \qquad \theta + g_+ L_0 \delta^3 < 1.$$

All bounds were fixed before this last reduction. On $r = kd^3$ the baseline successor has $d' > 0$, and

$$k(d')^3 - r' = k(a - 3bk)d^5 + O(d^6) > 0$$

after a further reduction of δ. For smaller r at the same s, isotonicity lowers both successor coordinates in (r, s); the allowed bound $k(\theta - s')^3$ increases as s' decreases. Thus $0 < r \le kd^3$ is a forward invariant noncrossing region.

On any noncrossing all-B orbit, $\sum_t r_t \le d_0/g_-$, so $r_t \to 0$. Consequently

$$r_0 = \sum_t r_t f(d_t) = \sum_t \frac{f(d_t)}{g(d_t)}(d_t - d_{t+1}) \le L_0 d_0^3. \tag{6}$$

The initial state with $r_0 = Md_0^3 < r_*$ must therefore cross. The set of noncrossing slice points is downward closed by comparison. It is relatively closed: a finite strict crossing is open in initial conditions, and a hit of $d = 0$ is followed by a strict crossing since $r > 0$ and $g(0) > 0$. Its maximal point $q(d_0)$ is attained and positive. If its limiting deficit were $d_\infty > 0$, then $r_t \to 0$ would force eventual entry into the strict interior $r < kd^3$ of the invariant region. Continuity of finitely many updates would allow a slightly larger noncrossing initial first coordinate, contradicting maximality. Hence its limiting deficit is zero.

For two ordered noncrossing orbits starting on the same slice with distinct r, their s gap becomes positive after the first update. Subsequently

$$s_{t+1}^{\mathrm{hi}} - s_{t+1}^{\mathrm{lo}} \geq (1 - L_g r_t^{\mathrm{hi}})(s_t^{\mathrm{hi}} - s_t^{\mathrm{lo}}).$$

The product of these factors is positive, because $L_g r_* < 1$ and $\sum_t r_t^{\mathrm{hi}} < \infty$. Both orbits cannot tend to $s = \theta$. This proves uniqueness and also that every smaller slice point has a positive limiting deficit. Estimate (6) gives the bound on q. In particular, a baseline successor of a critical point is the critical point on its new slice. No graph regularity is used. □

Lemma 4 (Polynomial drift)**.** *There is a polynomial $E(r, d) = rH(d)+P(d)$, with $H(d) = 1+O(d^2)$ and $P(d) = -ad^3/(3b) + O(d^4)$, such that on $0 \leq d \leq \rho$, $0 < r \leq L_0\rho^3$,*

$$|E(B(r, d)) - E(r, d)| \leq C_1 r\rho^6. \tag{7}$$

The expression $B(r, d)$ here uses the uncapped local formula, including a possible negative successor deficit. Along a switched segment from a critical state of deficit ρ up to its first crossing,

$$\sum_{t<\tau} |\Delta_B E(r_t, d_t)| \leq C_2\rho^7. \tag{8}$$

The analogous bound holds for every finite noncrossing segment and every infinite noncrossing tail.

Proof. Write the Taylor polynomials as $f_5 = ad^2 + cd^3 + ud^4 + vd^5$, $g_3 = b + ed + jd^2 + kd^3$. Take $H = 1 + k_2d^2 + k_3d^3$ and $P = \sum_{i=3}^{6} v_i d^i$, with

$$\begin{aligned}
k_2 &= -a/2, & k_3 &= (ae - 3bc)/(6b),\\
v_3 &= -a/(3b), & v_4 &= (ae - bc)/(4b^2),\\
v_5 &= (a^2b^2 + 2abj - 2ae^2 - 2b^2u + 2bce)/(10b^3),\\
v_6 &= (-2a^2b^2e + 3ab^3c + 3ab^2k - 6abej + 3ae^3\\
&\qquad - 3b^3v + 3b^2cj + 3b^2eu - 3bce^2)/(18b^4).
\end{aligned}$$

These symbols denote Taylor coefficients, not derivatives without their factorials. Expanding in r, the coefficients of r and r^2 in the polynomial baseline drift are respectively

$$-f_5H - g_3P', \qquad -(1 - f_5)g_3H' + \tfrac{1}{2}g_3^2P''.$$

Substitution makes these $O(d^6)$ and $O(d^3)$. All remaining terms are polynomials divisible by r^3. Thus the drift is $O(r\rho^6)$ on the stated region. This also follows from weight 3 for r, weight 1 for d, and cancellation of all terms of weight at most eight.

For the actual maps, $f - f_5 = O(d^6)$ and $g - g_3 = O(d^4)$. Their successor changes relative to the polynomial maps are $O(r\rho^6)$ in r and $O(r\rho^4)$ in d. The interpolating successors have $r' = O(\rho^3)$ and $|d'| = O(\rho)$, including at exit; reducing δ makes this immediate from $g_+L_0\delta^2 \leq 1$ and a fixed

bound on g_3. Here $E_r = O(1)$ and $E_d = O(\rho^2)$. The mean-value estimate gives (7). The polynomial E is defined also for negative d; no negative-d Taylor assumption on f or g is used.

On a switched segment from the indicated critical state, $r_t \le L_0\rho^3$ and $0 < d_t \le \rho$ at all prestates. The crossing overshoot satisfies $-g_+L_0\rho^3 \le d_\tau \le 0$, and the cap is inactive up to and including crossing by the radius choice above. Hence

$$\sum_{t<\tau} r_t \le \frac{\rho + g_+L_0\rho^3}{g_-} = O(\rho).$$

This estimate, which does not involve τ, proves (8). For noncrossing segments the same telescoping estimate has no overshoot. □

Lemma 5 (Critical rates)**.** *On any fixed critical baseline orbit,*

$$q(d) = \frac{a}{3b}d^3 + O(d^4), \qquad \sqrt{n}\,d_n \longrightarrow D, \qquad n^{3/2}r_n \longrightarrow \frac{aD^3}{3b}.$$

Proof. On an infinite critical tail E tends to zero. Lemma 4 therefore gives $|E(q(d), d)| \le C_2d^7$. Using $q(d) = O(d^3)$ in the polynomial formula gives the stated expansion. Thus $d_{n+1} = d_n - (a/3)d_n^3 + O(d_n^4)$, so $d_{n+1}^{-2} - d_n^{-2} = 2a/3 + O(d_n)$. Cesàro averaging gives $d_n^{-2}/n \to 2a/3$. Substitution yields the limit for r_n. These statements concern values of q, not its derivatives. □

4 From interventions to terminal values

Proof of Theorem 1. At the first A, the successor has the same s as the baseline successor, but its first coordinate is larger by $\epsilon = rf(d) > 0$. It lies strictly above the critical slice endpoint. Lemma 3 therefore implies finite crossing for its baseline tail. Since $B \le A$ and both maps are isotone, every tail crosses no later than that baseline tail. Positivity of r at finite times and terminal freezing give $R > 0$. This proves crossing independently of the energy estimate.

At each prestate, the two local successor deficits coincide. Affinity of E in r gives the exact identity

$$E(A(r, d)) - E(B(r, d)) = rf(d)H(d - g(d)r). \tag{9}$$

The initial energy is $O(\rho^7)$ by the preceding critical-tail argument. At exit, $R = O(\rho^3)$ and $|d_\tau| = O(\rho^3)$, so $E(R, d_\tau) = R + O(\rho^9)$. Summing (9), and treating the baseline drift hypothetically even at A prestates, gives

$$R = S + O(\rho^7) + O(\rho^2 S).$$

Lemma 4 makes the constants independent of schedule and crossing time. This proves (4). □

For a first intervention after n baseline updates, order comparison shows that the all-B tail minimizes every later state and hence the margin. On that tail $S = r_nf(d_n) = O(d_n^5)$, and therefore

$$2M_n = r_nf(d_n) + O(d_n^7). \tag{10}$$

Lemma 5 implies $n^{5/2}r_nf(d_n) \to a^2D^5/(3b)$, proving the first asymptotic in Theorem 2.

5 Exact ordering under a run constraint

Lemma 6 (Finite-word interchange)**.** *For every N, the length-N prefix of p_K produces a state no greater coordinatewise than any length-N word with no K consecutive B's.*

Proof. We have $B \le A$ and

$$A \circ B \le B \circ A. \tag{11}$$

Indeed $W(r,s) \ge s$, so the first coordinate of $A \circ B$ is $r[1 - F(\theta - s)]$, no greater than $r[1 - F(\theta - W(r,s))]$, the first coordinate of $B \circ A$. Their second coordinates are ordered by isotonicity of W in r. The argument includes crossing and capping because F is nondecreasing and vanishes at nonpositive arguments.

Transform a legal word to the periodic prefix from left to right. At a first mismatch, the target cannot be A: that position follows $K-1$ already matched B's, which force A in any legal word. The target is thus B and the current symbol is A. If a later B exists, move its first occurrence left across the intervening A's. Each replacement of a temporal AB by BA lowers the resulting state by (11); isotonicity preserves the inequality through the rest of the word. The moved B has an A before it until its last move. At that last move the already matched preceding run has length less than $K-1$. The run left at its old location shortens. Thus legality is preserved. If no later B exists, replace the current A by B; the following symbols are all A and the same preceding-run bound ensures legality. This replacement also lowers the state. Iteration terminates at the desired prefix. □

Apply this lemma to the prefix of each admissible infinite schedule at every N. Taking limits of the first coordinates proves $P_K \le m(\omega; z_0)$, while p_K is itself admissible. Thus $G_K = P_K$ for every $K \ge 2$.

6 The complete periodic tail

Let $t_j = jK - 1$ be the prestate time of the jth A under p_K. Set $\epsilon_j = r_{t_j} f(d_{t_j})$ before crossing and $\epsilon_j = 0$ after crossing. Then $S = \sum_{j \ge 1} \epsilon_j$. Prior to crossing, r_t and d_t decrease, so their product $r_t f(d_t)$ decreases. Each ϵ_j with $j > 1$ is bounded above by each of the preceding $K-1$ baseline losses. These disjoint blocks give, for every $J \ge 1$,

$$\sum_{j>J} \epsilon_j \le \frac{r_{t_J}}{K-1}. \tag{12}$$

The estimate remains valid if crossing truncates the nonzero kicks. At $t_1 = K-1$ the trajectory is exactly critical. Writing $\rho = d_{K-1} \sim DK^{-1/2}$, we obtain

$$S \le \epsilon_1 + \frac{r_{t_1}}{K-1} = O(K^{-5/2}) = O(\rho^5).$$

Theorem 1, applied from that prestate, gives

$$2P_K = S + O(\rho^7). \tag{13}$$

The omitted prefix does not change the margin, because r decreases and s increases throughout.

For completeness we prove the needed fixed-block limit for the switched orbit. Use the scaled time $x = 1 + (t - t_1)/K$ and variables $u = K^{3/2} r$, $z = K^{1/2} d$. At t_1, the initial values tend to

$(aD^3/(3b), D)$ by Lemma 5. For each fixed integer $J \geq 1$, the baseline recurrence on a bounded stopped region is

$$u^+ = u - \frac{auz^2}{K} + O(K^{-3/2}), \qquad z^+ = z - \frac{bu}{K} + O(K^{-3/2}).$$

At an A update the first equation omits its $O(K^{-1})$ loss; there are at most J such updates on $1 \leq x \leq J$. The limiting ODE and solution are

$$\dot{u} = -auz^2, \quad \dot{z} = -bu, \qquad u(x) = \frac{aD^3}{3b}x^{-3/2}, \quad z(x) = Dx^{-1/2}.$$

On a compact neighborhood of this solution the vector field is Lipschitz. Euler comparison gives

$$e_{m+1} \leq (1 + L_J/K)e_m + C_J K^{-3/2} + (C_J/K)\mathbf{1}_{\{A \text{ update}\}},$$

with the ODE local truncation error $O(K^{-2})$ absorbed. Discrete Gronwall over at most JK steps yields $\max e_m = O(e_0 + K^{-1/2}) = o(1)$. Stop provisionally at the first crossing or exit from that neighborhood. Its boundary can be chosen a positive distance from the limiting path, and $z(x) \geq D/\sqrt{J} > 0$; the estimate therefore excludes either stop for all sufficiently large K. This argument only fixes J; it assumes no uniformity as J grows. Consequently, for every fixed j,

$$K^{5/2}\epsilon_j \longrightarrow \frac{a^2D^5}{3b}j^{-5/2}, \qquad K^{3/2}r_{t_j} \longrightarrow \frac{aD^3}{3b}j^{-3/2}. \tag{14}$$

By (12) and (14),

$$\limsup_{K\to\infty} K^{5/2}\sum_{j>J}\epsilon_j \leq \frac{aD^3}{3b}J^{-3/2} \longrightarrow 0.$$

Truncate at J, pass $K \to \infty$, then let $J \to \infty$. Equations (13)–(14) yield

$$K^{5/2}P_K \to C\sum_{j\geq 1} j^{-5/2} = C\zeta(5/2).$$

Together with the exact optimizer, this completes Theorem 2.

7 Examples and limitations

The prototype is $f(d) = ad^2$, $g(d) = b$. Its polynomial energy simplifies to

$$E = r - \frac{a}{3b}d^3 - \frac{a}{2}rd^2 + \frac{a^2}{10b}d^5,$$

and gives $q(d) = ad^3/(3b) + O(d^5)$. A nonconstant example is $f(d) = d^2(1 + d)$, $g(d) = 1 + d$ for $d \geq 0$, with $\bar{g} = 1$ for $d < 0$ and any fixed $0 < \theta < 1$. For a sufficiently small rectangle it satisfies all hypotheses and has $a = b = 1$, hence the same leading constant as the corresponding prototype. More generally, higher coefficients in f and g may vary arbitrarily, subject to the stated regularity and a sufficiently small local domain.

The continuation is part of the theorem. Freezing preserves a positive crossing value; it does not itself force crossing or determine the leading value. To see the limitation of local information, for $\lambda > 0$ set

$$\widetilde{B}(r,s) = \left(\frac{r[1 - F(\theta - s)]}{1 + \lambda r}, W(r,s)\right), \qquad \widetilde{A}(r,s) = \left(\frac{r}{1 + \lambda r}, W(r,s)\right).$$

These maps remain isotone: $r \mapsto r/(1+\lambda r)$ is increasing, $1 - F(\theta - s)$ is nonnegative and nondecreasing in s, and W is unchanged. With weight 3 for r and weight 1 for d, their pre-crossing expansions retain the leading weighted contraction ard^2 and accumulation br. The added $O(r^2)$ loss has weight six, above the weight-five contraction, and is smaller on $r = O(d^3)$. The ordinary Taylor jet does change; the comparison here concerns these weighted terms. Nevertheless every schedule satisfies

$$r_{t+1} \le \frac{r_t}{1+\lambda r_t}, \qquad r_t \le \frac{1}{r_0^{-1} + \lambda t} \longrightarrow 0.$$

All margins are then zero. This is a counterexample to an extension based only on leading weighted local terms and monotonicity, not to the theorem: it violates terminal freezing and the exact omitted-contraction form.

Quadratic flatness is also structural. A perturbation adding a linear term to f is outside the hypotheses. The exponent $5/2$ therefore does not revise the exponent of any linear-loss model. The exact critical initialization is essential to the stated asymptotics; we have not proved uniform results for approximate critical states, parameter limits, or noisy recurrences. Adaptive choices between the two specified modes are covered pathwise: each resulting trajectory has a schedule to which the theorems apply, with their stated intervention and run conditions. Interventions that change the update maps or inject disturbances are outside the scope. The arbitrary-tail transfer bound permits S larger than $O(\rho^5)$, as occurs for repeated A updates. Only the single-intervention and sparse periodic schedules use the sharper remainder.

Reproducibility and review status

The package includes `validation/verify_final.py`, a coefficient snapshot, and the validation receipts. The script matches the six printed energy coefficients to the snapshot, checks nine exact drift cancellations and the scaled ODE identities, and exhaustively checks the finite-word transformation for lengths at most twelve and $2 \le K \le 5$. Its citation-key, label, delimiter, and environment checks are syntactic; source attribution is assessed separately in the literature-review record. A workspace-only option checks preservation hashes; the mathematical and source checks can also run on the standalone package. These computations supplement the analytic proof and are not formal verification. No numerical approximation to a critical state is used as proof.

This manuscript was prepared with AI assistance. The accompanying records identify two independent AI reviews of the complete supplied source, including a hostile review, and a separate scoped editorial/literature review. They are not human peer review or external endorsement. Compilation and page-by-page visual inspection are documented in the accompanying production receipt.

References

[1] W. T. Jamieson and O. Merino, Local dynamics of planar maps with a non-isolated fixed point exhibiting 1–1 resonance, *Advances in Difference Equations* (2018), Article 142. doi:10.1186/s13662-018-1595-x.

[2] C. Cufí-Cabré and E. Fontich, Differentiable invariant manifolds of nilpotent parabolic points, *Discrete and Continuous Dynamical Systems* **41**(10) (2021), 4667–4704. doi:10.3934/dcds.2021053.

[3] D. Angeli and E. D. Sontag, Monotone control systems, *IEEE Transactions on Automatic Control* **48**(10) (2003), 1684–1698. doi:10.1109/TAC.2003.817920.

[4] H. E. Lomelí, J. D. Meiss and R. Ramírez-Ros, Canonical Melnikov theory for diffeomorphisms, *Nonlinearity* **21** (2008), 485–508. doi:10.1088/0951-7715/21/3/007.