

# Successive Concept Differentiation as Observation Refinement

Carrier Splitting, Interaction-Core Homology, and Descent

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## Abstract

A 2015 proposal for concept unfolding combined Boolean distinction, simplicial boundary, and local-to-global reconstruction. We separate these operations and study one precise refinement model: a new Boolean observation splits the states of a finite observational poset  $Q$ , producing an induced subposet  $Q^\psi \subseteq Q \times \{0 < 1\}$  with projection  $\pi$ .

Carrier splitting factors through a homotopy-neutral split completion followed by order thinning. In weighted height at most two its cone has a complete integral rooted-incidence classification. At arbitrary height, a two-colour missing-copy model identifies the obstruction. If the exclusive inversion graph is a matching, cone homology is the direct sum of coarse edge-link reduced homology shifted by two, with no torsion-freeness assumption.

Beyond matching, a filtration by mixed-core size has first page given by available-link homology. When cores have at most three vertices, an explicit chain mapping cone computes the defect and the spectral sequence collapses at  $E^2$ , with integral extensions retained. Maximum-degree-two exclusive graphs stabilize at  $E^3$ ; an eight-state  $K_{2,2}$  example proves this bound sharp. A seven-state star rules out unconditional  $E^2$  collapse for forests, Ferrers graphs, and chordal-bipartite graphs. An all-exclusive star family further shows that interaction can create torsion even when all local links have torsion-free homology.

The Boolean historical reconstruction and cover/descent framework are retained as separate interpretations. The general cofiber, obstruction-complex, spectral-sequence, nerve, and sheaf machinery is classical. The contribution claimed here is the explicit observation-product specialization and its positive classifications and integral failure boundaries; publication priority for the exact specialization remains subject to review.

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# 1 Purpose and scope

The motivating question is

Can successive differentiation of a concept be represented rigorously as a transformation between complexes?

The answer developed here is yes, provided that four levels are not conflated:

- (i) a semantic state space and its attributes;
- (ii) an observation or distinction process;
- (iii) a combinatorial/topological representation of the observed structure;
- (iv) a genuine chain differential on that representation.

The historical thesis is used only as a source of ideas. The mathematical claims below stand on newly stated definitions.

The resulting architecture has two primary operators:

$$d : C_n(K_k; R) \longrightarrow C_{n-1}(K_k; R), \quad d^2 = 0,$$

and

$$U_k : C_\bullet(K_k; R) \longrightarrow C_\bullet(K_{k+1}; R),$$

where  $d$  is the ordinary chain differential and  $U_k$  is induced by an unfolding/refinement map. When  $U_k$  is a chain map, it induces homology maps; when the represented support changes,  $H_*(\text{Cone}(U_k))$  measures the change relative to the chosen representation.

## 1.1 Conventions and result hierarchy

Finite observational posets are assumed nonempty. An order complex includes the empty face. Its augmented chain complex is denoted  $\tilde{C}_*$ ; in particular  $\tilde{H}_{-1}(\Delta\emptyset; R) = R$ . Joins and suspensions of empty links use this simplicial convention; the formal  $(-1)$ -sphere has no vertices. Ordinary homology is zero in negative degrees. A coefficient ring is commutative with identity.

For a chain map  $f : A_* \rightarrow B_*$ , use  $\text{Cone}(f)_n = B_n \oplus A_{n-1}$  and  $d(b, a) = (d_B b + f(a), -d_A a)$ . For a map of spaces or simplicial complexes,  $\text{Cone}(f)$  denotes its homotopy cofiber and its homology is explicitly reduced. For nonempty source and target these conventions agree under the usual chain comparison. A simplicial map that repeats an image vertex sends that oriented simplex to zero on normalized chains. A shift  $C[k]_n = C_{n-k}$  has differential  $(-1)^k d_C$ .

The main mathematical results are organized as follows.

- (1) Theorem 7.2 converts a changing observational carrier into a neutral split completion followed by order thinning.
- (2) Theorem 7.9 gives the exact integral low-height result; Theorem 8.3 gives the all-height coarse-edge-link formula.
- (3) Theorems 9.2 and 9.3 describe controlled overlap, including the sharp distinction between  $E^2$  and  $E^3$  collapse.

- (4) Theorem 11.2 treats cover refinement and descent separately. It reconstructs data on the support actually covered; restriction from  $X$  to a proper subspace  $Y$  need not preserve or uniquely extend global sections.

The historical Boolean and simplex-skeleton constructions motivate the model; they are not a claim to new general Boolean, polyhedral, or sheaf theory.

## 2 Historical operator: what survives and what does not

### 2.1 Fixed-factor unfolding

Let  $A = \{a_1, \dots, a_m\}$  be a finite attribute set and let  $\mathbf{B}(A)$  be the Boolean functions on  $\{0, 1\}^A$ .

**Definition 2.1** (Factorized concept). *A factorized concept is an ordered tuple  $F = (f_1, \dots, f_m)$  with  $f_i \in \mathbf{B}(A)$ . Its conjunction semantics is*

$$\Pi(F) = \bigwedge_{i=1}^m f_i.$$

**Definition 2.2** (Historical one-step unfolding). *For a chosen factorization  $F = (f_1, \dots, f_m)$ , define*

$$U_{\text{th}}(F) = \bigvee_{j=1}^m \left( \neg f_j \wedge \bigwedge_{i \neq j} f_i \right).$$

**Proposition 2.3** (Exact-one-false characterization). *At a valuation  $x$ ,  $U_{\text{th}}(F)(x) = 1$  if and only if exactly one factor value  $f_i(x)$  is false.*

*Proof.* The  $j$ th disjunct requires  $f_j(x) = 0$  and every other factor to be 1. Distinct disjuncts are mutually exclusive. Hence the disjunction is true exactly on the one-false layer.  $\square$

For the positive conjunction  $X_1 \wedge \dots \wedge X_m$ , the one-step output is therefore the Hamming layer with exactly one zero. This is a precise mathematical version of “one further distinction.”

### 2.2 Representation dependence

**Theorem 2.4** (Non-congruence of factorized unfolding). *Fix the arity  $m = |A| \geq 2$ . There is no Boolean-function operator  $\bar{U} : \mathbf{B}(A) \rightarrow \mathbf{B}(A)$  satisfying*

$$\bar{U}(\Pi(F)) = U_{\text{th}}(F)$$

*for every  $m$ -factor tuple  $F \in \mathbf{B}(A)^m$ .*

*Proof.* The constant Boolean functions belong to  $\mathbf{B}(A)$ . At the fixed arity  $m$ , take

$$F = (0, 1, 1, \dots, 1), \quad F' = (0, 0, 1, \dots, 1).$$

Both conjunctions are the zero function. Exactly one factor is false in  $F$ , so  $U_{\text{th}}(F) = 1$ ; exactly two are false in  $F'$ , so  $U_{\text{th}}(F') = 0$ . One operator on the conjunction cannot give both.  $\square$

For  $m = 1$ , by contrast,  $\Pi((f)) = f$  and  $U_{\text{th}}((f)) = \neg f$ ; ordinary negation is such an operator. Thus the failure of semantic descent begins at arity two. For  $m \geq 2$ , either internal factorization/provenance is part of the concept, a canonical normal form must be chosen, or the historical rule must be replaced by a semantic operator.

### 2.3 Boolean derivative interpretation

For the monomial  $f(x) = \prod_i x_i$ , the Boolean partial derivative

$$\partial_j^B f = f(x_j = 0) \Downarrow f(x_j = 1)$$

is  $\prod_{i \neq j} x_i$ . Therefore

$$U_{\text{th}}(f) = \bigvee_j \left( (1 - x_j) \wedge \partial_j^B f \right).$$

The historical rule is thus a negation-gated Boolean sensitivity operator, not the simplicial boundary.

## 3 What the attached “Logical Complexes” notes recover

The attached notes contain a useful idea, but two algebraic levels must be separated.

### 3.1 XOR is a difference vector, not a scalar distance

For Boolean state vectors  $x, y \in \mathbb{F}_2^m$ ,

$$x \Downarrow y = x + y$$

is the vector difference in characteristic two. The scalar Hamming distance is

$$d_H(x, y) = \text{wt}(x \Downarrow y).$$

Thus expressions such as “XOR is the distance” should be replaced by: XOR is the coordinatewise disagreement vector, whose Hamming weight is the distance.

### 3.2 The three-variable tetrahedron is genuine

Set

$$G = (1, 1, 1), \quad D = (1, 1, 0), \quad E = (1, 0, 1), \quad F = (0, 1, 1).$$

Then

$$D \Downarrow G = (0, 0, 1), \quad E \Downarrow G = (0, 1, 0), \quad F \Downarrow G = (1, 0, 0).$$

These three differences form a basis of  $\mathbb{F}_2^3$ ; viewed as points in  $\mathbb{R}^3$ , the corresponding real differences are also linearly independent. Hence  $\{G, D, E, F\}$  is an affine 3-simplex, a tetrahedron. The four points are not redundant.

A low-arity coincidence in the notes is also correct:

$$D \Downarrow E = F,$$

and cyclically. Here  $\{0, D, E, F\}$  is the even-parity plane in  $\mathbb{F}_2^3$ . This closure does not persist for general  $m$ : XOR of two one-flip children of  $1^m$  has Hamming weight 2, not  $m - 1$ .

### 3.3 Why the signs disappear over $\mathbb{F}_2$

The notes also anticipate a valid characteristic-two phenomenon. If  $[v_0, \dots, v_n]$  is a formal simplex, then over  $\mathbb{F}_2$

$$d[v_0, \dots, v_n] = \sum_i [v_0, \dots, \widehat{v}_i, \dots, v_n]$$

contains no visible alternating signs because  $-1 = +1$ . Every codimension-two face appears twice in  $d^2$ , so  $d^2 = 0$ .

The important correction is that the addition above occurs in the *chain group on formal simplices*. It is not XOR of the bit-vector labels themselves. Label-space XOR and chain addition happen to use the same field in this model, but they are distinct operations on distinct vector spaces.

## 4 Exact unfolding layers and hypersimplices

Let the fully undifferentiated positive state be  $\mathbf{1} = (1, \dots, 1) \in \{0, 1\}^m$ . For  $S \subseteq [m]$ , define

$$x_S = \mathbf{1} \updownarrow \chi_S,$$

so precisely the coordinates in  $S$  are switched from 1 to 0.

**Definition 4.1** (Exact distinction layer). *The  $k$ th exact unfolding layer is*

$$L_k^{(m)} = \{x_S : |S| = k\}.$$

Throughout this section, the layer index is an integer satisfying  $0 \leq k \leq m$ .

**Theorem 4.2** (Boolean unfolding simplex and hypersimplex theorem). *For  $m \geq 2$ :*

- (i)  $|L_k^{(m)}| = \binom{m}{k}$  and every  $x \in L_k^{(m)}$  has Hamming weight  $m - k$ ;
- (ii)  $\{\mathbf{1}\} \cup L_1^{(m)}$  is an affine  $m$ -simplex;
- (iii) the convex hull of the exact  $k$ th layer is the hypersimplex

$$\text{conv}(L_k^{(m)}) = \left\{ x \in [0, 1]^m : \sum_i x_i = m - k \right\};$$

- (iv) for  $0 < k < m$ , this polytope has dimension  $m - 1$  and is a simplex exactly in the extremal cases  $k = 1$  or  $k = m - 1$ .

*Proof.* Part (i) follows by choosing the  $k$  zero coordinates. For (iii), the displayed intersection of the cube with the integer-sum hyperplane is a bounded polytope. If two coordinates of a point lie strictly between 0 and 1, perturb one upward and the other downward by a sufficiently small amount; the point is not a vertex. Exactly one fractional coordinate is impossible because the coordinate sum is an integer. Hence every vertex is a 0–1 vector with exactly  $k$  zero coordinates, proving the hull equality. For (ii), if  $u_i = \mathbf{1} \updownarrow e_i$ , then  $u_i - \mathbf{1} = -e_i$  in  $\mathbb{R}^m$ , so the  $m$  difference vectors are linearly independent. Part (iv) is the standard dimension and vertex description of the hypersimplex [16]; in the two extremal nontrivial weights its vertices are the standard simplex vertices up to complementation.  $\square$

For  $m = 4$  the exact layers have the particularly transparent geometry

$$\text{point} \longrightarrow \text{tetrahedron} \longrightarrow \text{octahedron} \longrightarrow \text{tetrahedron} \longrightarrow \text{point},$$

since  $\Delta(4, 2)$  is an octahedron. Thus repeated unfolding does not naturally stay inside the category of simplices if one geometrizes each exact state layer by convex hull.

## 5 A canonical cumulative unfolding complex

There is, however, a canonical simplicial model of *distinction histories*. A subset  $S \subseteq [m]$  records the attributes that have been distinguished so far.

**Definition 5.1** (Cumulative distinction complex). *For  $0 \leq r \leq m-1$ , define*

$$K_r^{(m)} = \{S \subseteq [m] : 1 \leq |S| \leq r+1\}.$$

*Thus  $K_r^{(m)} = \text{sk}_r \Delta^{m-1}$ , the  $r$ -skeleton of the full simplex on the attribute set. We also retain the empty face implicitly.*

The inclusions

$$K_0^{(m)} \hookrightarrow K_1^{(m)} \hookrightarrow \dots \hookrightarrow K_{m-1}^{(m)} = \Delta^{m-1}$$

form a genuine filtration. Here increasing  $r$  means allowing larger jointly distinguished attribute sets.

**Theorem 5.2** (Exact cone profile of cumulative unfolding). *Let  $R$  be any coefficient ring and let*

$$i_r : K_{r-1}^{(m)} \hookrightarrow K_r^{(m)}, \quad 1 \leq r \leq m-1.$$

*Then*

$$H_q(\text{Cone}(C_*(i_r; R))) \cong H_q(K_r^{(m)}, K_{r-1}^{(m)}; R) \cong \begin{cases} R_{(r+1)}^{(m)}, & q = r, \\ 0, & q \neq r. \end{cases}$$

*Proof.* The relative chain complex  $C_*(K_r, K_{r-1}; R)$  has no new cells below degree  $r$  and has one degree- $r$  basis element for every  $(r+1)$ -subset of  $[m]$ . There are no relative cells above degree  $r$ . Hence every relative differential is zero and the displayed relative homology follows. The mapping cone of an inclusion computes the relative homology of the pair [7].  $\square$

**Corollary 5.3** (Absolute homology of intermediate distinction stages). *For  $0 \leq r < m-1$  and a field  $\mathbb{k}$ ,*

$$\tilde{H}_q(K_r^{(m)}; \mathbb{k}) = 0 \quad (q \neq r), \quad \dim \tilde{H}_r(K_r^{(m)}; \mathbb{k}) = \binom{m-1}{r+1}.$$

*The final stage  $K_{m-1}^{(m)} = \Delta^{m-1}$  is contractible.*

*Proof.* The augmented chain complex of the full simplex is exact. In each degree  $q < r$ , the skeleton has the same chain groups and adjacent boundary maps, so its reduced homology vanishes there. There are no chains above degree  $r$ . Its reduced Euler characteristic is

$$-1 + \sum_{j=0}^r (-1)^j \binom{m}{j+1} = (-1)^r \binom{m-1}{r+1}.$$

Thus its only possible reduced Betti number has the asserted rank. The final stage is the full simplex and is contractible.  $\square$

This gives a useful correction to the older suggestion that cycles should represent symmetry. In this exact model the transient cycles are forced combinatorially by missing higher-order fillers. They measure incomplete compatibility, not group-theoretic symmetry.

**Example 5.4** (Four attributes). *For  $m = 4$  the cumulative stages have Betti profiles*

$$K_0 : (\beta_0) = (4), \quad K_1 : (1, 3), \quad K_2 : (1, 0, 1), \quad K_3 : (1, 0, 0, 0).$$

*The relative/cone ranks at successive steps are 6 in degree 1, 4 in degree 2, and 1 in degree 3, exactly the numbers of newly admitted edges, triangles, and tetrahedra.*

## 6 Observation systems and three kinds of refinement

The word “refinement” is overloaded. Three operations should be distinguished.

**Definition 6.1** (Signature refinement). *An observation map is  $q_k : X \rightarrow O_k$ . We say  $q_{k+1}$  refines  $q_k$  when there is  $r_k : O_{k+1} \rightarrow O_k$  with  $q_k = r_k \circ q_{k+1}$ . Equivalently, every  $q_{k+1}$ -fiber lies in a  $q_k$ -fiber.*

For finite propositional observation families, adding observables refines the generated topology, can split observational equivalence classes, thins the specialization preorder, and can increase signature distance. This is an observation-theoretic notion of differentiation.

A second regime occurs when a new observation is constant on every current  $T_0$  equivalence class. Then the carrier does not split; only order relations can be removed. This becomes a same-carrier order-thinning problem, which is treated separately in companion work. If the new observation separates previously identified worlds, the refined carrier gains points and a different construction is required.

A third notion, used in the main theorem below, is a *cover refinement*. It concerns local pieces representing a support, not merely enlargement of an observation vocabulary.

**Remark 6.2** (Presentation-dependent nerve warning). *For an indexed family of truth regions  $\{\varsigma(\varphi_i)\}$ , the observation-side nerve records joint satisfiability, but it is presentation-dependent. Adding a tautological observation can cone the nerve without changing the generated observation topology. Therefore arbitrary extension of an observation list should not be identified with a topologically meaningful nerve filtration. The main theorem uses actual refinement of covers with explicit containment data.*

## 7 Carrier splitting under a new Boolean observation

We now treat the refinement regime left open in the previous draft: a new observation is not constant on every current observational-equivalence class, so one old  $T_0$  state can split into two refined states. This operation is not a cover refinement and is not, on its face, a same-carrier order thinning. It does, however, admit a canonical factorization through one.

Let  $W$  be the finite world set, let  $\Phi$  be the current observation family, and write

$$Q = o_\Phi(W) \subseteq \{0, 1\}^I$$

for the finite  $T_0$  quotient ordered coordinatewise. Add one Boolean observation  $\psi$ . The refined quotient is

$$Q^\psi = o_{\Phi \cup \{\psi\}}(W) \subseteq Q \times \{0, 1\},$$

with the product order. Let

$$\pi : Q^\psi \longrightarrow Q, \quad \pi(s, a) = s,$$

be the coordinate-forgetting map. For  $a \in \{0, 1\}$  put

$$A_a := \{s \in Q : (s, a) \in Q^\psi\}, \quad C := A_0 \cap A_1.$$

Thus  $A_0 \cup A_1 = Q$ , and  $C$  is precisely the set of old observational states that split.

**Definition 7.1** (Split completion). *On the same carrier as  $Q^\psi$ , define the split-completion order  $\hat{Q}^\psi$  by*

$$(s, a) \leq_{\hat{Q}} (t, b) \iff (s < t \text{ in } Q) \text{ or } (s = t \text{ and } a \leq b).$$

Equivalently,  $\widehat{Q}^\psi$  is the lexicographic sum over  $Q$  of the nonempty fiber chains

$$F_s := \{a \in \{0, 1\} : (s, a) \in Q^\psi\}.$$

Let  $b(s, a) = a$  be the new-bit label on this expanded carrier.

The distinction between  $Q^\psi$  and  $\widehat{Q}^\psi$  is exact. The split completion remembers every old comparison between coarse states, independently of the new bit. The actual refined order keeps only those comparisons along which the new bit is nondecreasing.

**Theorem 7.2** (Carrier-splitting factorization). *With the notation above:*

- (i) *the projection  $p : \widehat{Q}^\psi \rightarrow Q$ ,  $p(s, a) = s$ , is a homotopy equivalence of finite posets; in fact  $\widehat{Q}^\psi$  beat-reduces to a subposet isomorphic to  $Q$ ;*
- (ii) *the actual refined quotient is exactly the one-bit thinning of the split completion,*

$$Q^\psi = (\widehat{Q}^\psi)_b,$$

*meaning that a comparison  $x <_{\widehat{Q}} y$  is retained precisely when  $b(x) \leq b(y)$ ;*

- (iii) *if  $i : \Delta(Q^\psi) \hookrightarrow \Delta(\widehat{Q}^\psi)$  is the inclusion, then  $\Delta(\pi) = \Delta(p) \circ i$ , and there is a homotopy equivalence of homotopy cofibers*

$$\mathrm{Cof}(\Delta\pi) \simeq \mathrm{Cof}(i).$$

*Consequently, for every coefficient ring  $R$ ,*

$$\widetilde{H}_n(\mathrm{Cone}(\Delta\pi); R) \cong H_n(\Delta(\widehat{Q}^\psi), \Delta(Q^\psi); R).$$

*Proof.* For every split state  $s \in C$ , the lower copy  $(s, 0)$  is an up-beat point of  $\widehat{Q}^\psi$ : its strict upper set has the minimum  $(s, 1)$ , because  $(s, 1)$  lies below every copy over every  $t > s$ . Remove these lower split copies one at a time. What remains has one representative over each  $s \in Q$  and is order-isomorphic to  $Q$ . Beat-point removal is a strong deformation retraction for finite  $T_0$  spaces [19], giving (i).

For distinct coarse states  $s < t$ , the split completion always contains  $(s, a) < (t, b)$  whenever those copies exist. The product order on  $Q^\psi$  retains exactly the cases  $a \leq b$ . The same statement holds within a split fiber, where  $(s, 0) < (s, 1)$  is retained. This proves (ii). Part (iii) follows from  $\pi = p \circ i$  and homotopy invariance of the homotopy cofiber under postcomposition by the homotopy equivalence  $\Delta(p)$ . Since  $i$  is a simplicial inclusion, its homotopy cofiber computes the relative homology of its simplicial pair.  $\square$

**Remark 7.3** (What is standard and what is specific). *Beat-point reduction is classical finite-space topology [19], and homotopy cofiber invariance is standard; Quillen-type poset equivalences even admit simple-homotopy refinements in the standard setting [20]. The project-specific content is the observation-theoretic factorization: an actual split of  $T_0$  observational states is converted canonically into a homotopy-neutral lexicographic expansion followed by the precise one-bit monotonicity thinning already studied on a fixed carrier. Thus carrier splitting is not a fourth unrelated operation; it is a neutral blow-up plus an order defect.*

## 7.1 The exact inversion-chain complex

Theorem 7.2 gives a chain-level classification without any nerve hypothesis.

**Theorem 7.4** (Inversion-chain model). *For any coefficient ring  $R$ , the relative group*

$$C_k(\Delta(\widehat{Q}^\psi), \Delta(Q^\psi); R)$$

*is free on the strict chains*

$$x_0 <_{\widehat{Q}} x_1 <_{\widehat{Q}} \cdots <_{\widehat{Q}} x_k$$

*whose bit word*

$$b(x_0)b(x_1) \cdots b(x_k)$$

*is not weakly increasing. Equivalently, a relative generator is exactly a split-completion chain containing an inversion  $1 \cdots 0$ . Its relative boundary is the ordinary alternating simplicial boundary with every monotone-bit face set to zero.*

*Proof.* The two order complexes have the same vertex set. A split-completion chain is a simplex of  $\Delta(Q^\psi)$  exactly when every comparison in the chain survives the one-bit thinning, which is equivalent to the bit word being weakly increasing. Quotienting the simplicial chain groups therefore leaves exactly the nonmonotone chains, and the relative differential is the induced simplicial differential.  $\square$

This formulation is useful computationally: the defect depends only on the old signature poset and the three-valued fiber profile

$$F_s \in \{\{0\}, \{1\}, \{0, 1\}\},$$

not on the number of worlds inside each old equivalence class.

## 7.2 Immediate homotopy-neutrality criteria

The factorization also reveals when splitting is automatically conservative.

**Proposition 7.5** (Monotone-section criterion). *If  $A_1$  is an upward-closed subset of  $Q$ , then  $\pi : Q^\psi \rightarrow Q$  is a homotopy equivalence. Dually, if  $A_0$  is downward closed, then  $\pi$  is a homotopy equivalence.*

*Proof.* Assume  $A_1$  is upward closed. Define

$$r_+(s) = \begin{cases} (s, 1), & s \in A_1, \\ (s, 0), & s \notin A_1. \end{cases}$$

The upset condition makes  $r_+$  order preserving, and  $\pi r_+ = \text{id}_Q$ . Moreover every  $(s, a) \in Q^\psi$  satisfies  $(s, a) \leq r_+(s)$ , so  $\text{id}_{Q^\psi} \leq r_+ \pi$ . The order-homotopy lemma gives  $r_+ \pi \simeq \text{id}$  and hence  $\pi$  is a homotopy equivalence. The  $A_0$  statement is dual, using the minimum available bit in each fiber.  $\square$

**Corollary 7.6** (Common-outcome anchor). *If every old observational state has at least one refined world with  $\psi = 1$  ( $A_1 = Q$ ), or every old state has at least one with  $\psi = 0$  ( $A_0 = Q$ ), then carrier splitting is homotopy-neutral. In particular, if every old state genuinely splits, so  $A_0 = A_1 = Q$ , then  $Q^\psi = Q \times \{0 < 1\}$  and the projection is a homotopy equivalence.*

Thus the act of duplicating a state is not itself the source of topological defect. The defect is caused by incompatible outcome availability across comparable coarse states: a lower state that can realize 1 together with an upper state that can realize 0 creates a potentially deleted comparison.

### 7.3 Localization by lower fibers

For  $s \in Q$  define the lower fiber

$$L_s := \pi^{-1}(Q_{\leq s}).$$

If  $s \in A_1$ , then  $(s, 1)$  is a maximum of  $L_s$ , hence  $L_s$  is contractible. Therefore all lower-fiber obstructions to Quillen-type equivalence localize to the 0-only states  $A_0 \setminus A_1$ . Dually, all upper-fiber obstructions localize to the 1-only states  $A_1 \setminus A_0$ .

This observation does not replace the exact relative complex above. It does, however, connect the carrier-splitting problem to the classical poset-fiber framework. Under the connectivity hypotheses of the Björner–Wachs–Welker fiber theorem [21], the source  $\Delta(Q^\psi)$  admits a wedge decomposition in terms of the noncontractible lower fibers and upper links. The important point here is that the split fibers themselves disappear from that list whenever their top copy is present.

### 7.4 Exact low-height classification

The general inversion complex can have arbitrarily high dimension. In the first nontrivial dimensional regime it collapses to graph incidence and admits a closed integral classification.

**Definition 7.7** (Weighted split height). *Let  $C = A_0 \cap A_1$  be the split set. Define*

$$h_\psi(Q) := \max_{s_0 < \dots < s_r \text{ in } Q} (r + \#\{j : s_j \in C\}).$$

**Lemma 7.8.**  $h_\psi(Q) = \dim \Delta(\hat{Q}^\psi)$ .

*Proof.* A strict coarse chain contributes one vertex from every fiber, and a split fiber can contribute both copies consecutively. Taking both copies at every split state on the chain attains the displayed number, and no split-completion chain can contain more.  $\square$

Suppose  $h_\psi(Q) \leq 2$ . An *inversion edge* is a deleted order-complex edge of  $\Delta(\hat{Q}^\psi)$ , equivalently a pair

$$(s, 1) <_{\hat{Q}} (t, 0), \quad s < t,$$

with  $s \in A_1$  and  $t \in A_0$ . A deleted triangle has one of the four nonmonotone bit patterns

$$010, \quad 100, \quad 101, \quad 110.$$

It therefore contains either one or two inversion edges. Introduce one formal root for each connected component of  $Q$  (equivalently of  $\hat{Q}^\psi$ ). The *carrier-split defect graph*  $\mathcal{D}_\psi(Q)$  has one vertex for every inversion edge; a deleted triangle with two inversion edges joins their vertices, while a deleted triangle with one inversion edge joins that vertex to the root of its component. Parallel graph edges are allowed. Here a rooted forest means a forest with exactly one formal root in each graph component. This incidence graph  $\mathcal{D}_\psi$  is different from the exclusive inversion graph introduced later.

**Theorem 7.9** (Carrier-splitting cone classification in weighted height two). *Assume  $h_\psi(Q) \leq 2$ . Let  $c_0(\mathcal{D}_\psi)$  be the number of connected components of  $\mathcal{D}_\psi(Q)$  containing no formal root. Then*

$$\tilde{H}_n(\text{Cone}(\Delta\pi); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\beta_1(\mathcal{D}_\psi)}, & n = 2, \\ \mathbb{Z}^{c_0(\mathcal{D}_\psi)}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

*The relative boundary matrix is totally unimodular. Hence the same Betti ranks hold over every field. Moreover the following are equivalent:*

- (i) the carrier-splitting cone is acyclic over  $\mathbb{Z}$ ;
- (ii) it is acyclic over every field;
- (iii)  $\mathcal{D}_\psi(Q)$  is a rooted forest;
- (iv)  $\Delta(\widehat{Q}^\psi)$  collapses onto  $\Delta(Q^\psi)$  through deleted edge-triangle pairs;
- (v)  $\Delta\pi : \Delta(Q^\psi) \rightarrow \Delta(Q)$  is a homotopy equivalence.

*Proof.* The relative complex has only  $C_2 \rightarrow C_1$ , since all vertices are retained and the split completion has dimension at most two. A deleted triangle of bit type 100 or 110 has two inversion edges with opposite boundary signs; a triangle of type 010 or 101 has one. Consequently its matrix is the oriented incidence matrix of  $\mathcal{D}_\psi$  with formal-root rows deleted. Each graph component contains at most one root, because triangles cannot connect different coarse components. Relative graph homology therefore gives  $H_2 = \mathbb{Z}^{\beta_1(\mathcal{D}_\psi)}$  and  $H_1 = \mathbb{Z}^{c_0(\mathcal{D}_\psi)}$ . Incidence matrices and their row submatrices are totally unimodular.

Vanishing of both groups is exactly the rooted-forest condition. In a rooted tree with an edge, choose a nonroot leaf. Its inversion edge occurs in just one remaining deleted triangle and in no retained triangle, so the pair is an elementary simplicial collapse. Iterating leaf removal deletes every relative edge and triangle. Conversely a relative collapse gives a homotopy equivalence and hence vanishing relative homology. Finally the split-completion projection is a homotopy equivalence, so two-out-of-three gives the statement about  $\Delta\pi$ . The free groups displayed above also show the equivalence with acyclicity over every field.  $\square$

**Remark 7.10** (Status of the low-height theorem). *The graph-incidence algebra, total unimodularity, rooted forests, and beat-point technology are standard. The same height-two one-bit result is developed in the companion order-thinning manuscript [25]; the proof above makes the transfer used here self-contained. The new point here is the carrier-splitting reduction that makes that theorem applicable to a process which changes the  $T_0$  carrier. Accordingly, this theorem should be positioned as an exact transfer theorem, not as a new general theory of rooted forests.*

## 7.5 Sharpness beyond weighted height two

No coefficient-independent forest criterion can hold in unrestricted dimension. The carrier-splitting family contains the no-split case  $C = \emptyset$ , which is exactly arbitrary same-carrier one-bit thinning. Companion results [25] show that one-bit thinning in height three can already have relative integral torsion, and in unrestricted height can encode arbitrary finite-poset homotopy types inside a contractible coarse cone. If one insists that at least one old state genuinely split, a disjoint neutral split component may be adjoined without changing positive-degree cone homology. Thus the weighted-height-two hypothesis in Theorem 7.9 marks a real boundary rather than a presentational convenience.

The appropriate all-height object is therefore the inversion-chain complex of Theorem 7.4, supplemented where useful by poset-fiber or homotopy-colimit methods. The next two sections give all-height results under matching and interaction-core hypotheses, together with explicit obstructions to stronger graph-class conclusions.

## 8 All-height carrier splitting under noninteracting exclusive inversions

The low-height rooted-incidence theorem gives a complete answer when the weighted height is at most two. In higher dimensions a universal rooted-forest criterion is impossible: the same-carrier boundary case already supports integral torsion and arbitrary finite-poset homotopy types. Nevertheless, an all-height exact classification survives when the genuinely incompatible carrier outcomes do not interact.

Let

$$A_a = \{s \in Q : (s, a) \in Q^\psi\}, \quad a \in \{0, 1\},$$

and put

$$P = A_1 \setminus A_0, \quad Z = A_0 \setminus A_1, \quad C = A_0 \cap A_1.$$

Thus  $P$  consists of the 1-only coarse states,  $Z$  of the 0-only coarse states, and  $C$  of the states that genuinely split.

**Definition 8.1** (Exclusive inversion graph). *The exclusive inversion graph is the bipartite graph*

$$G_\psi = (P \sqcup Z, E_\psi), \quad E_\psi = \{(p, z) \in P \times Z : p < z \text{ in } Q\}.$$

*An edge records a coarse comparison whose earlier endpoint admits only the new outcome 1 while its later endpoint admits only 0.*

### 8.1 The full-doubling model

Let

$$D = Q \times \{0 < 1\}, \quad K = \Delta D, \quad L = \Delta Q^\psi.$$

The actual refined carrier is the induced subposet of  $D$  obtained by deleting exactly

$$M_0 = \{(p, 0) : p \in P\}, \quad M_1 = \{(z, 1) : z \in Z\}.$$

Let  $K_P$  be obtained by deleting  $M_0$ , let  $K_Z$  be obtained by deleting  $M_1$ , and put

$$U = K_P \cup K_Z, \quad L = K_P \cap K_Z.$$

The complex  $K_P$  closes upward to the full top row, while  $K_Z$  closes downward to the full bottom row. Hence both  $K_P \hookrightarrow K$  and  $K_Z \hookrightarrow K$  are homotopy equivalences. Since  $H_*(K, K_P) = H_*(K, K_Z) = 0$ , the triple  $(K, K_P, L)$ , simplicial excision for  $U = K_P \cup K_Z$ , and the triple  $(K, U, K_Z)$  give  $H_n(K, L) \cong H_n(K_P, L) \cong H_n(U, K_Z) \cong H_{n+1}(K, U)$ . Thus

$$H_n(K, L; R) \cong H_{n+1}(K, U; R) \tag{1}$$

for every coefficient ring  $R$ . Since  $D \rightarrow Q$  is a homotopy equivalence,  $H_n(K, L; R)$  is the reduced homology of the homotopy cofiber of the refined projection  $\Delta\pi : \Delta Q^\psi \rightarrow \Delta Q$ .

A simplex of  $K \setminus U$  contains at least one missing lower copy  $(p, 0)$  and at least one missing upper copy  $(z, 1)$ . Because it is a product-order chain, every such pair satisfies  $p < z$ . Therefore the missing vertices of a relative simplex project to a complete bipartite subgraph of  $G_\psi$ . This observation isolates the source of higher interaction: bicliques with more than one edge couple otherwise local defects.

**Proposition 8.2** (Empty-exclusive-graph neutrality). *If  $E_\psi = \emptyset$ , then  $\Delta\pi$  is a homotopy equivalence.*

*Proof.* Set  $b(s) = 1$  exactly when some  $p \in P$  satisfies  $p \leq s$ . This isotone choice takes an available value at every state: it is 1 on  $P$  and cannot be 1 on  $Z$  without an exclusive edge. The section  $r(s) = (s, b(s))$  obeys  $\pi r = \text{id}$ . On  $Q^\psi$ , the map  $c(s, a) = (s, \max\{a, b(s)\})$  is available and isotone, with  $\text{id} \leq c \leq r\pi$ . Comparable poset maps are homotopic, so  $r$  is a homotopy inverse.  $\square$

**Theorem 8.3** (Exclusive-Inversion Matching Theorem). *Suppose that the exclusive inversion graph  $G_\psi$  is a matching. Then, for every coefficient ring  $R$  and every integer  $n$ ,*

$$\tilde{H}_n(\text{Cone}(\Delta\pi); R) \cong \bigoplus_{(p,z) \in E_\psi} \tilde{H}_{n-2}(\text{lk}_{\Delta Q}[p, z]; R). \quad (2)$$

*Consequently  $\Delta\pi$  is an  $R$ -homology equivalence if and only if every coarse edge link  $\text{lk}_{\Delta Q}[p, z]$  indexed by an exclusive inversion is  $R$ -acyclic.*

*Proof.* A relative simplex of  $(K, U)$  contains at least one vertex of  $M_0$  and one vertex of  $M_1$ . If it contained two missing lower vertices and one missing upper vertex, the corresponding upper endpoint would have degree at least two in  $G_\psi$ ; the dual statement holds for two missing upper vertices. The matching hypothesis therefore forces every relative simplex to contain a unique mixed pair

$$e = ((p, 0) < (z, 1)), \quad (p, z) \in E_\psi.$$

Deleting either endpoint of  $e$  sends the simplex into  $U$ , while deleting any other vertex preserves the same pair. Reorienting each simplex with the two core vertices first gives an even chain shift. Hence the relative chain complex splits as

$$C_m(K, U; R) = \bigoplus_{e \in E_\psi} C_m^{(e)}, \quad C_m^{(e)} \cong \tilde{C}_{m-2}(\text{lk}_K e; R).$$

Thus

$$H_m(K, U; R) \cong \bigoplus_{e \in E_\psi} \tilde{H}_{m-2}(\text{lk}_K e; R).$$

It remains to identify the product-order link. For  $e = ((p, 0) < (z, 1))$ , its link factors into the lower interval, the open middle interval, and the upper interval. The open middle interval is covered by two contractible subposets: delete  $(z, 0)$  and close upward to the bit-1 row, or delete  $(p, 1)$  and close downward to the bit-0 row. Their intersection is  $(p, z)_Q \times \{0 < 1\}$ , homotopy equivalent to  $(p, z)_Q$ . Since  $(p, 1)$  and  $(z, 0)$  are incomparable, every middle-chain lies in one of these two pieces; therefore

$$\Delta((p, 0), (z, 1)) \simeq \Sigma \Delta(p, z)_Q.$$

Joining with the lower and upper factors gives

$$\text{lk}_{\Delta(Q \times \mathbf{2})} e \simeq \Sigma \text{lk}_{\Delta Q}[p, z].$$

Combining this suspension shift with (1) yields (2).  $\square$

**Corollary 8.4** (Bounded-poset localization). *Assume that  $Q$  has a global minimum  $\hat{0}$  and maximum  $\hat{1}$ , and that  $G_\psi$  is a matching. Then every exclusive edge except possibly  $(\hat{0}, \hat{1})$  has contractible coarse edge link. Hence only the global endpoint inversion can contribute to the cone homology.*

*Proof.* If  $p \neq \hat{0}$ , the lower factor  $\Delta Q_{<p}$  has the minimum  $\hat{0}$ ; if  $z \neq \hat{1}$ , the upper factor  $\Delta Q_{>z}$  has the maximum  $\hat{1}$ . In either case the edge link is a join with a nonempty contractible complex.  $\square$

**Corollary 8.5** (Distributive endpoint profile). *Let  $Q = J(R)$  be the lattice of order ideals of a nonempty finite poset  $R$ . Make the bottom state 1-only, the top state 0-only, and every other state split. If  $R$  is an  $r$ -element antichain, so that  $Q = B_r$ , then*

$$\tilde{H}_r(\text{Cone}(\Delta\pi); \mathbb{Z}) \cong \mathbb{Z}$$

*and all other reduced cone homology vanishes. If  $R$  is not an antichain, the cone is integrally acyclic.*

*Proof.* Corollary 8.4 reduces the answer to the reduced homology of the proper part of  $J(R)$ , shifted by two. By the classical crosscut theorem [26], the atom crosscut is a full simplex unless the atoms join to the top element. In  $J(R)$  the atoms correspond to the minimal elements of  $R$ , and they join to the top exactly when every element of  $R$  is minimal, i.e. exactly when  $R$  is an antichain. In that case  $J(R) = B_r$  and its proper part is homotopy equivalent to  $S^{r-2}$ .  $\square$

## 8.2 Sharpness: interaction begins at the first shared endpoint

The matching condition cannot be weakened merely to “ $G_\psi$  is a forest.” For the three-element chain  $0 < 1 < 2$  with availability profile

$$0 : \{1\}, \quad 1 : \{0\}, \quad 2 : \{0\},$$

the exclusive graph is the two-edge star  $K_{1,2}$ . Direct integral Smith normal form gives

$$\tilde{H}_1(\text{Cone}(\Delta\pi); \mathbb{Z}) \cong \mathbb{Z},$$

while the sum of the two individual coarse-edge-link contributions is zero. The shared endpoint creates a three-vertex interaction core and a differential between the two edge defects.

Nor does matching imply total unimodularity or torsion-freeness. Given any nonempty finite poset  $R$ , take

$$Q = \hat{0} \oplus R \oplus \hat{1}$$

and use the endpoint profile above. There is exactly one exclusive inversion, and Theorem 8.3 gives

$$\tilde{H}_n(\text{Cone}(\Delta\pi); \mathbb{Z}) \cong \tilde{H}_{n-2}(\Delta R; \mathbb{Z}).$$

Thus arbitrary finite-poset homology, including torsion, can occur even inside the matching class. In particular, total unimodularity must be controlled by stronger conditions on the relevant links; global torsion-freeness alone is not a TU criterion [24].

The matching hypothesis is sufficient, not necessary: the same proof applies whenever there is no mixed three-vertex core. Section 9 makes this criterion and the nonmatching interaction maps explicit.

**Remark 8.6** (Priority boundary). *The deletion/cofiber technology used here is classical. In particular, vertex-cover obstruction complexes already provide general homotopical tools for comparing a complex with unions of induced subcomplexes [23], while poset-fiber decompositions give broad results under fiber-connectivity hypotheses [21]. The candidate contribution of Theorem 8.3 is the observation-theoretic two-colour recognition and the resulting exact coarse edge-link formula. No claim is made that simplicial vertex deletion or homotopy cofibers themselves are new.*

## 9 Interaction cores beyond matching

Retain  $K = \Delta(Q \times \mathbf{2})$ ,  $L = \Delta Q^\psi$ , the missing sets  $M_0 = P \times \{0\}$  and  $M_1 = Z \times \{1\}$ , and  $U = K_P \cup K_Z$ . Cone homology in this section is reduced homology of the topological homotopy cofiber, equivalently homology of the algebraic mapping cone. The two-colour shift is

$$\tilde{H}_n(\text{Cone}(\Delta\pi); R) \cong H_{n+1}(K, U; R).$$

For a mixed missing core  $S$ , put  $\Lambda_S = \text{lk}_K(S)[V(L)]$ . All link chain complexes below are augmented, including degree  $-1$ .

**Proposition 9.1** (Interaction spectral sequence). *The finite-filtration construction [8] gives a convergent spectral sequence*

$$E_{p,q}^1 = \bigoplus_{|S|=p} \tilde{H}_q(\Lambda_S; R) \implies H_{p+q}(K, U; R),$$

with  $d^r$  of bidegree  $(-r, r-1)$ . The first differential is the signed sum of the inclusion maps obtained by deleting one missing vertex, omitting nonmixed faces. If all mixed cores have size at most  $M \geq 2$ , then  $E^{M-1} = E^\infty$ .

*Proof.* Orient a relative simplex by listing its  $p$  missing vertices first and its available vertices second, with the corresponding shuffle sign. Its boundary is

$$d(S \wedge T) = \sum_i (-1)^i (S \setminus \{s_i\}) \wedge T + (-1)^p S \wedge \partial T.$$

Terms with nonmixed core vanish in the relative complex. Filtering by  $p$  gives the displayed page and differential. The filtration is finite and exhaustive. The abutment is the associated graded of homology, so integral extension data may remain. Only columns  $2, \dots, M$  can occur, so every  $d^r$  with  $r \geq M-1$  has zero source or target.  $\square$

**Theorem 9.2** (Exact two-column interaction theorem). *Suppose no mixed core has four or more vertices. Define*

$$A_\bullet = \bigoplus_{|S|=2} \tilde{C}_\bullet(\Lambda_S; R), \quad B_\bullet = \bigoplus_{|S|=3} \tilde{C}_\bullet(\Lambda_S; R),$$

and let  $h : B_\bullet \rightarrow A_\bullet$  be the signed sum of core-deletion inclusions. Then

$$C_\bullet(K, U; R) \cong \text{Cone}(h)[2], \quad \tilde{H}_n(\text{Cone}(\Delta\pi); R) \cong H_{n-1}(\text{Cone}(h)).$$

The interaction spectral sequence collapses at  $E^2$ , and there is a natural exact sequence

$$0 \longrightarrow \text{coker } H_{n-1}(h) \longrightarrow \tilde{H}_n(\text{Cone}(\Delta\pi); R) \longrightarrow \ker H_{n-2}(h) \longrightarrow 0.$$

In particular the cone is  $R$ -acyclic if and only if  $h$  is a quasi-isomorphism.

*Proof.* The degree- $m$  relative group is  $A_{m-2} \oplus B_{m-3}$  and its boundary is  $(a, b) \mapsto (\partial a + hb, -\partial b)$ . This is the asserted shifted chain cone. Apply the two-colour shift, the mapping-cone long exact sequence, and Proposition 9.1 with  $M = 3$ .  $\square$

The exact sequence splits noncanonically over a field. Over  $\mathbb{Z}$  it splits when the right-hand kernel is free abelian; no splitting is asserted in general. Thus collapse alone does not replace the integral answer by a direct sum of link-homology kernels and cokernels. Signed incidence coefficients on link chains likewise do not imply total unimodularity of the induced maps on homology.

**Theorem 9.3** (Degree-two classification and its sharp page bound). *Suppose the exclusive graph has maximum degree at most two. Every mixed core has size at most four, so  $E^3 = E^\infty$ . If no  $C_4$  component has both its same-colour pairs comparable in  $Q$ , Theorem 9.2 applies and  $E^2 = E^\infty$ . In every cycle component of length at least six the cone contribution is the coarse-edge-link direct sum of the matching theorem.*

*Proof.* Transitivity gives  $N(p') \subseteq N(p)$  when  $p < p'$  in  $P$ , and  $N(z) \subseteq N(z')$  when  $z < z'$  in  $Z$ . A mixed core is a biclique with each colour class a chain. Degree two bounds both class sizes by two. A four-core requires a whole  $K_{2,2}$  component with both colour pairs comparable. This proves the bounds. On a longer cycle, two same-colour degree-two vertices have distinct neighbourhoods of the same cardinality and cannot be comparable if they share a neighbour. Hence there are no triple cores. No missing vertex can then lie in a product-edge link, so the original matching proof applies componentwise without change.  $\square$

More explicitly, a path with at least three edges supports triples only on its first two or last two edges: a comparable common-neighbour pair must include a leaf. For a path of at least four edges these end triples have disjoint edge supports, giving independent star blocks and interior edge blocks. A three-edge path has up to two triples sharing its middle edge and is treated by one instance of  $\text{Cone}(h)$ . Every surviving relative face remains in the same exclusive graph component, so the component decomposition is at chain level. Links must nevertheless be calculated in the whole coarse poset.

For a two-edge star with ordered leaves,  $h$  is the signed pair of inclusions from its triple link to its two edge links. On the frozen three-chain profile  $(1, 0, 0)$ , all three available links have reduced  $H_0 = \mathbb{Z}$  and  $h_0$  is the primitive signed diagonal  $\mathbb{Z} \rightarrow \mathbb{Z}^2$ . This gives cone  $H_1 = \mathbb{Z}$ , although both coarse edge links are contractible.

## 9.1 Higher differentials are necessary

For a  $C_4$  component whose two pairs are comparable, restrict to its relative chain summand and write the core order  $a < b < x < y$ . Let  $A, B, V$  be the sums of edge, triple, and four-core link chains respectively. Then

$$C_m(K, U) = A_{m-2} \oplus B_{m-3} \oplus V_{m-4}, \quad d(a, b, c) = (\partial a + hb, -\partial b + kc, \partial c),$$

where  $h, k$  are the signed core-deletion inclusions and  $hk = 0$ . In triple order  $(abx, aby, axy, bxy)$ , the signs of  $k$  are  $(-, +, -, +)$ , while  $h(abx) = bx - ax$ ,  $h(aby) = by - ay$ ,  $h(axy) = ax - ay$ , and  $h(bxy) = bx - by$ , each term meaning its corresponding inclusion. The only possible higher differential is

$$d^2 : E_{4,q}^2 \longrightarrow E_{2,q+1}^2.$$

If  $\partial c = 0$  and  $\partial b = kc$ , it is represented by  $hb$ . Thus a differential lowering core size by at most one can still produce a nonzero secondary differential.

An explicit eight-state example makes this bound sharp. Take  $Q = A_2 \oplus R \oplus A_2$ , with bottom antichain  $\{a, a'\}$ , top antichain  $\{y, y'\}$ , and  $R = \{v, b, w, x\}$  with relations  $v < x$ ,  $b < w$ ,  $b < x$ . Make  $a, b$  one-only and  $x, y$  zero-only; split the other states. The only nonzero first-page groups are  $E_{2,1}^1 = \mathbb{Z}$  and  $E_{4,0}^1 = \mathbb{Z}$ . The four-core link consists of  $b_1, x_0$ . The triple links  $abx$  and  $bxy$  are cones with apex  $y'_1$  and  $a'_0$ , respectively; the links  $aby$  and  $axy$  are the two paths displayed below. The edge links  $ax$ ,  $by$ , and  $bx$  are cones, with apex  $y'_1$ ,  $a'_0$ , and either of those vertices. The remaining edge link  $ay$  has maximal simplices

$$[a_1, b_1, w_1], \quad [v_0, x_0, y_0], \quad [a_1, v_1], \quad [v_0, v_1], \quad [w_0, w_1], \quad [w_0, y_0].$$

Collapsing its two triangle ears leaves a cycle. Thus the stated first-page groups follow directly from all nine available links. The paths

$$x_0 - v_0 - v_1 - a_1 - b_1, \quad x_0 - y_0 - w_0 - w_1 - b_1$$

in the two relevant triple links fill the four-core difference. Give each path orientation from  $x_0$  to  $b_1$ . In the  $ay$  component of the signed zigzag, the first path minus the second traverses the loop once; the other edge components are cycles in contractible links. Therefore  $d^2$  is a unit  $\mathbb{Z} \rightarrow \mathbb{Z}$ , and  $E^3 = 0$ . The supplementary signed-chain certificate verifies these fillings and the integral homology directly.

Nor do Ferrers or chordal-bipartite graphs force  $E^2$  collapse. The seven-state poset with covers  $0 < 1$ ,  $0 < 2$ ,  $1 < 3$ ,  $2 < 3$ ,  $2 < 5$ ,  $3 < 4$ ,  $3 < 6$ ,  $5 < 6$ , and profile  $(1, 2, 0, 0, 2, 2, 0)$  has exclusive graph  $K_{1,3}$ . Its integral  $E^1 = E^2$  has  $\mathbb{Z}^2$  in bidegree  $(2, 1)$  and  $\mathbb{Z}$  in  $(4, 0)$ ;  $d^2$  is a primitive injection. Its only nonzero cone group is  $H_2 = \mathbb{Z}$ . Here profile value 2 means split. The four-core link is  $\{0_1, 2_0\} \simeq S^0$ . The three triple links, indexed by their upper missing states 023, 026, and 036, contain respective paths

$$A : 2_0 - 4_1 - 0_1, \quad B : 2_0 - 5_1 - 0_1, \quad C : 2_0 - 3_0 - 1_0 - 1_1 - 0_1.$$

All three triple links are contractible. The 02 edge link is the square on bipartition  $\{0_1, 2_0\} \sqcup \{4_1, 5_1\}$ ; the 03 edge link is a cone with apex  $4_1$ . The 06 edge link has maximal faces

$$[0_1, 1_1], [0_1, 5_1], [1_0, 1_1], [1_0, 3_0, 6_0], [2_0, 3_0, 6_0], [2_0, 5_0, 5_1], [2_0, 5_0, 6_0].$$

Collapsing its triangle ears leaves the cycle  $0_1 - 1_1 - 1_0 - 3_0 - 2_0 - 5_1 - 0_1$ . Orient  $A, B, C$  from  $2_0$  to  $0_1$ . With the missing-core deletion signs, their horizontal boundary has components  $B - A$  in the 02 link,  $A - C$  in the 03 link, and  $C - B$  in the 06 link. The first traverses the 02 square once; hence  $d^2 : \mathbb{Z} \rightarrow \mathbb{Z}^2$  has a unit coordinate and is primitive. This gives a signed witness independent of the certificate's rank calculation. For the abutment, write  $T = Q \setminus \{0\}$  and  $C = \{1, 4, 5\}$ . The complex  $\Delta T$  is contractible: its two maximal-element lower ideals are cones whose intersection is the lower ideal of 3. The complex  $\Delta C$  is an edge and an isolated point. In a star profile  $Q = \{p\} \oplus T$ , with  $p$  one-only,  $C$  split, and  $T \setminus C$  zero-only, the refined complex is the union of  $\Delta((T \times \{0\}) \cup (C \times \{1\}))$  and the cone  $\Delta((\{p\} \times \{1\}) \cup (C \times \{1\}))$ . The first piece retracts to  $\Delta T$ , and the intersection is  $\Delta C$ . The homotopy-pushout description consequently gives  $\tilde{H}_n(\text{Cone}(\Delta\pi)) \cong H_{n-1}(\Delta T, \Delta C)$ . Here this is  $\mathbb{Z}$  only at  $n = 2$ ; the sole possible  $d^2$  must be a primitive injection. The supplementary certificate records all link matrices and signed chains. These are explicit counterexamples, not global minimality claims.

## 9.2 An unbounded interacting class and new torsion

**Proposition 9.4** (All-exclusive star). *Let  $R$  be a nonempty finite poset. Take  $Q = \{p\} \oplus R$ , make  $p$  one-only and every state of  $R$  zero-only. Then  $E^2 = E^\infty$  integrally and*

$$\tilde{H}_n(\text{Cone}(\Delta\pi); \mathbb{Z}) \cong H_{n-1}(\Delta R; \mathbb{Z}),$$

where the right side is ordinary homology.

*Proof.* Every core is  $\{p_0\} \cup \sigma_1$  for a nonempty chain  $\sigma$  in  $R$ . Its available link is the isolated vertex  $p_1$  disjoint from the cone  $\Delta(R_{\leq \min \sigma} \times \{0\})$ . The difference-of-components generators identify all nonzero link homology with  $\mathbb{Z}$  in degree zero and identify their inclusion maps with the identity. Consequently  $d^1$  is, up to a uniform sign, the ordinary simplicial boundary of  $\Delta R$ . The single row collapses on  $E^2$  and has only one filtration grade in each total degree.  $\square$

For the nonempty-face poset of a triangulated projective plane, all these available links have torsion-free homology, and every coarse edge link is a vertex link in its barycentric subdivision, hence a circle. Nevertheless the interaction differential creates  $\mathbb{Z}/2$  in cone degree two. The explicit six-vertex triangulation in the supplement gives a 32-state example, with every link and the reduced interaction complex checked by integral Smith normal form.

### 9.3 Whole-relation Horn closure is neutral

**Proposition 9.5.** *If the nonempty refined Boolean relation is closed under coordinatewise meet, then  $\Delta\pi$  is a homotopy equivalence. The dual statement holds for coordinatewise join closure.*

*Proof.* Finiteness and meet closure give a minimum in the refined relation. Its coordinate projection is again meet closed and nonempty, so also has a minimum. Both order complexes are contractible. Join closure gives maxima.  $\square$

The hypothesis concerns the whole realized refined relation; it is not merely a closure assumption on a selected subset of observations. This is an elementary application of closure-system topology, not a new Horn theorem.

**Remark 9.6** (Scope and priority). *Filtered-complex spectral sequences, link/cofiber recursions, and their extension issues are classical. The complexes  $\Lambda_S$  are precisely vertex-cover obstruction complexes in the sense of Chachólski–Jin–Scolamiero–Tombari [23]. The results here explicitly identify which such complexes interact in the observation product model, with positive class restrictions and integral obstructions. They do not introduce a new general obstruction theory or supersede poset-fiber theorems.*

## 10 Cover refinement and canonical homology maps

**Definition 10.1** (Nerve). *After discarding empty members of a finite cover  $\mathcal{U} = \{U_i\}_{i \in I}$  of  $X$ , its nerve  $\text{Nrv}(\mathcal{U})$  has vertex set  $I$  and simplex  $\sigma \subseteq I$  exactly when*

$$U_\sigma := \bigcap_{i \in \sigma} U_i \neq \emptyset.$$

**Definition 10.2** (Refinement map). *Let  $\mathcal{V} = \{V_j\}_{j \in J}$  and  $\mathcal{U} = \{U_i\}_{i \in I}$  cover the same space, or let the union of  $\mathcal{V}$  lie in the union of  $\mathcal{U}$ . A refinement choice is a map  $\rho : J \rightarrow I$  such that*

$$V_j \subseteq U_{\rho(j)}$$

*for every  $j$ .*

**Lemma 10.3** (Contiguity of refinement choices). *Every refinement choice induces a simplicial map*

$$N(\rho) : \text{Nrv}(\mathcal{V}) \rightarrow \text{Nrv}(\mathcal{U}).$$

*Any two refinement choices  $\rho, \rho'$  are contiguous and therefore induce the same homomorphism on homology.*

*Proof.* If  $\sigma \in \text{Nrv}(\mathcal{V})$ , choose  $x \in \bigcap_{j \in \sigma} V_j$ . Then  $x$  belongs to every  $U_{\rho(j)}$  and every  $U_{\rho'(j)}$ . Hence the union of the two image vertex sets spans a simplex of  $\text{Nrv}(\mathcal{U})$ , which is precisely contiguity.  $\square$

This lemma is standard. In particular, Leitão’s 2026 framework [14] for persistent homology of cover refinements makes this contiguity principle explicit at the cover level. It should not be presented here as a new theorem.

## 11 Observation–Refinement Descent and Cone Theorem

We now state the central structural result in a form adapted to successive concept differentiation.

**Definition 11.1** (Pair-good subcomplex cover). *Let  $Y \subseteq X$  be a finite simplicial pair. A finite family  $\mathcal{U} = \{U_i\}_{i \in I}$  of subcomplexes covering  $X$  is good relative to  $Y$  if every nonempty intersection  $U_\sigma = \bigcap_{i \in \sigma} U_i$  is contractible and every nonempty  $U_\sigma \cap Y$  is contractible. The restricted cover  $\mathcal{U}|_Y = \{U_i \cap Y\}_{i \in I}$  is indexed after deleting its empty members. A finite subcomplex cover of  $Y$  is good if all its nonempty intersections are contractible. The subcomplex condition is part of these definitions, not a consequence of intersection contractibility alone.*

**Theorem 11.2** (Observation–Refinement Descent and Cone Theorem). *Let  $Y \subseteq X$  be a finite simplicial pair. Let  $\mathcal{U}$  be a finite pair-good subcomplex cover of  $X$  and let  $\mathcal{V}$  be a finite good subcomplex cover of  $Y$  refining  $\mathcal{U}|_Y$ . Delete empty indexed members before taking nerves. Thus all three covers  $\mathcal{U}$ ,  $\mathcal{U}|_Y$ , and  $\mathcal{V}$  lie in the functorial subcomplex-cover nerve theorem. Choose any refinement map*

$$\rho : \mathcal{V} \longrightarrow \mathcal{U}|_Y$$

and let

$$u : \mathrm{Nrv}(\mathcal{V}) \xrightarrow{N(\rho)} \mathrm{Nrv}(\mathcal{U}|_Y) \hookrightarrow \mathrm{Nrv}(\mathcal{U})$$

be the induced simplicial map. Then:

- (a) *the homotopy class up to contiguity, and hence the induced homology map  $u_*$ , is independent of the refinement choice;*
- (b)  *$N(\rho) : \mathrm{Nrv}(\mathcal{V}) \rightarrow \mathrm{Nrv}(\mathcal{U}|_Y)$  is a homotopy equivalence;*
- (c) *for any coefficient ring  $R$ ,*

$$H_n(\mathrm{Cone}(C_*(u; R))) \cong H_n(X, Y; R)$$

*naturally up to the standard nerve identifications;*

- (d) *in the conservative case  $Y = X$ , the cone is acyclic, equivalently  $u$  is a quasi-isomorphism;*
- (e) *assume additionally that  $X$ ,  $Y$ , the cover members, and their intersections are admissible objects of a site with inclusion morphisms for the displayed subcomplexes, their pullbacks are represented by these intersections, and the displayed families are covers in that site. Take the sheaf and cosheaf values in a category admitting the finite limits and colimits used below, for example abelian groups. For a sheaf  $\mathcal{F}$ , compatible data on  $\mathcal{U}$  reconstruct  $\mathcal{F}(X)$ ; compatible data on  $\mathcal{U}|_Y$  or  $\mathcal{V}$  reconstruct  $\mathcal{F}(Y)$ . Refinement between the latter two covers preserves that section on  $Y$ . No unique extension from  $Y$  to  $X$  is asserted;*
- (f) *under the same site hypothesis, a cosheaf  $\mathcal{G}$  is assembled as the colimit of the intersection diagram on each covered support: the colimits are  $\mathcal{G}(X)$  and  $\mathcal{G}(Y)$ , respectively. Refinement on  $Y$  preserves its colimit; inclusion induces the comparison  $\mathcal{G}(Y) \rightarrow \mathcal{G}(X)$ , which need not be an isomorphism.*

*Proof.* Part (a) is Lemma 10.3. For (b), the finite subcomplex-cover nerve theorem [27, 15] supplies compatible equivalences for  $\mathcal{V}$  and  $\mathcal{U}|_Y$  with  $Y$ . Functoriality for the refinement map identifies  $N(\rho)$  with the identity of  $Y$  through these equivalences. Hence  $N(\rho)$  is a homotopy equivalence.

Let  $j : \text{Nrv}(\mathcal{U}|_Y) \hookrightarrow \text{Nrv}(\mathcal{U})$ . Since  $N(\rho)$  is a homotopy equivalence, the mapping cones of  $j \circ N(\rho)$  and  $j$  have isomorphic homology; this follows, for example, by comparing the long exact sequences of the two cones and applying the five lemma. The mapping cone of the chain inclusion  $C_*(j)$  computes

$$H_n(\text{Nrv}(\mathcal{U}), \text{Nrv}(\mathcal{U}|_Y); R).$$

The relative nerve theorem identifies this with  $H_n(X, Y; R)$ , proving (c). If  $Y = X$ , the relative groups vanish, giving (d).

For (e), apply the sheaf axiom [9] separately on each support: compatible families on a cover of  $S$  are the equalizer describing  $\mathcal{F}(S)$ , for  $S = X$  or  $S = Y$ . The covers  $\mathcal{U}|_Y$  and  $\mathcal{V}$  cover the same  $Y$ , so their gluing maps agree with refinement. Restriction  $\mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  is a separate map and need not be bijective. Part (f) is the dual cosheaf colimit statement, again on each support separately.  $\square$

**Example 11.3** (Restriction is not unique extension). *Take the discrete space  $X = \{a, b\}$ ,  $Y = \{a\}$ , and the sheaf of integer-valued functions. Singleton covers satisfy the stated good-cover conditions. Data on  $Y$  determine an element of  $\mathbb{Z}$ , whereas  $\mathcal{F}(X) = \mathbb{Z}^2$ ; restriction is the projection onto the first coordinate. A section on  $Y$  therefore has many extensions to  $X$ . This distinguishes descent on a fixed support from an extension problem.*

## 11.1 What the theorem means for unfolding

The theorem separates two questions that were previously mixed:

- (i) *Does the chosen map fail to be a homology equivalence?* Cone/relative homology detects this failure; its vanishing alone does not detect every change of support or homotopy type.
- (ii) *Can compatible local descriptions be reconstructed globally?* This is answered by descent or colimit conditions.

A finer description can therefore be topologically conservative while still providing more local information. Conversely, one may have perfect local gluing on a support whose topology has genuinely changed.

## 12 Descent defects for data that are not already a sheaf

If one assumes a sheaf from the beginning, reconstruction is built into the axioms. For modeling it is useful to measure how a candidate local-data system fails to be a sheaf.

Let  $\mathcal{F}$  be a presheaf of abelian groups and  $\mathcal{U} = \{U_i\}$  a finite cover. Define

$$C^0(\mathcal{U}; \mathcal{F}) = \prod_i \mathcal{F}(U_i), \quad C^1(\mathcal{U}; \mathcal{F}) = \prod_{i < j} \mathcal{F}(U_i \cap U_j),$$

and

$$(\delta^0 s)_{ij} = s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}.$$

Let  $Z^0 = \ker \delta^0$  be the compatible local families and let

$$\text{res} : \mathcal{F}(X) \rightarrow Z^0(\mathcal{U}; \mathcal{F})$$

be global restriction.

**Definition 12.1** (Descent defects). *Define*

$$D_{\text{uniq}}(\mathcal{U}; \mathcal{F}) = \ker(\text{res}), \quad D_{\text{exist}}(\mathcal{U}; \mathcal{F}) = \text{coker}(\text{res}).$$

**Proposition 12.2** (Exact reconstruction criterion). *Compatible local data reconstruct a unique global element if and only if*

$$D_{\text{uniq}} = 0 = D_{\text{exist}}.$$

*In particular both defects vanish for a sheaf.*

This motivates a two-axis defect signature for an unfolding step  $u$ :

$$\mathfrak{D}(u; \mathcal{F}) = (H_*(\text{Cone } C_*(u)), D_{\text{uniq}}, D_{\text{exist}}).$$

The first entry is structural; the latter two are semantic/local-to-global. They should not be collapsed into one number.

## 13 Exact simplicial gluing and the no-filler warning

**Proposition 13.1** (Exact gluing as a colimit). *If a simplicial complex  $K$  is covered by subcomplexes  $\{K_i\}_{i \in I}$ , then  $K$  is the colimit of the full diagram consisting of the  $K_i$  and all their finite intersections, with the evident inclusion maps.*

This is the rigorous combinatorial version of rebuilding an object from pieces with overlap provenance.

**Proposition 13.2** (Bare proper faces do not determine a filler). *The proper-face data of an  $n$ -simplex do not by themselves determine whether the  $n$ -simplex is present.*

*Proof.* The boundary complex  $\partial\Delta^n$  and the filled simplex  $\Delta^n$  have exactly the same proper faces, but only the latter contains the top-dimensional simplex.  $\square$

Therefore the historical idea of an “integrator” requires a filler rule, provenance, a flag/cliue convention, a colimit diagram containing the intended top cell, or semantic descent data. Conjunction of face-formulas is not a general simplicial gluing operation.

## 14 Finite prototypes

### 14.1 Three attributes: vertices, cycle, filler

The cumulative distinction filtration for  $m = 3$  is

$$K_0 = \{1, 2, 3\} \hookrightarrow K_1 = \partial\Delta^2 \hookrightarrow K_2 = \Delta^2.$$

Over  $\mathbb{F}_2$ :

$$H_0(K_0) \cong \mathbb{F}_2^3, \quad H_1(K_1) \cong \mathbb{F}_2, \quad H_{>0}(K_2) = 0.$$

The first step adds three edges and has

$$H_1(K_1, K_0) \cong \mathbb{F}_2^3.$$

The second adds one triangle and has

$$H_2(K_2, K_1) \cong \mathbb{F}_2.$$

The one-dimensional cycle at the middle stage is therefore not a symmetry invariant; it is the obstruction created by having all pairwise compatibilities without the triple filler.

## 14.2 Four attributes: a tetrahedral history complex

For  $m = 4$ :

$$K_0 \subset K_1 \subset K_2 \subset K_3 = \Delta^3,$$

where  $K_1$  is the complete graph  $K_4$  and  $K_2$  is the tetrahedral boundary. The nonzero reduced homology is

$$\tilde{H}_0(K_0) \cong R^3, \quad H_1(K_1) \cong R^3, \quad H_2(K_2) \cong R.$$

The next stage fills the tetrahedron and kills the 2-cycle.

At the exact-state level, however, the  $k = 2$  Hamming layer has six points and convex hull an octahedron. This illustrates why the state-layer geometry and the distinction-history complex are related but not identical constructions.

## 15 Symmetry, indistinguishability, and homology are different invariants

Let  $\text{Aut}(K)$  be the simplicial automorphism group. Let an observation map  $q : X \rightarrow O$  define indistinguishability by equal fibers. Neither is determined by homology.

A filled triangle has automorphism group  $S_3$  but trivial positive-dimensional homology. Conversely, an asymmetric unicyclic graph can have  $H_1 \cong \mathbb{Z}$  and trivial automorphism group. Thus cycles cannot generally be identified with symmetry.

There is nevertheless a rigorous symmetry-breaking statement at the observational level. If a group  $G$  acts on  $X$ , define

$$G_q = \{g \in G : q(gx) = q(x) \ \forall x \in X\}.$$

If  $q_{k+1}$  refines  $q_k$ , then

$$G_{q_{k+1}} \leq G_{q_k}.$$

Finer observation can therefore break observational symmetry by shrinking the subgroup that preserves all observations. This is separate from the homology of the chosen complex.

## 16 Relation to observation topologies and order thinning

The observation-topology program provides a natural semantic base for the present construction. An indexed family  $\Phi = (\varphi_i)$  defines signatures and an observation-generated topology. Extending  $\Phi$  by new coordinates makes the topology finer, refines observational equivalence, and thins the specialization preorder. These facts supply a mathematically clean interpretation of “more differentiated observation.”

However, the passage from observation topology to complex must be chosen explicitly. The nerve of the indexed truth regions records joint satisfiability but is not an invariant of the generated topology. This is why the present paper uses a cover-refinement contract when applying the nerve theorem.

There is also an important same-carrier subcase. If a new Boolean observation factors through the current  $T_0$  quotient, it does not split observational states; it only removes comparisons incompatible with the new bit. Companion work shows that, for a height-two current signature poset and one new bit, homotopy neutrality can be recognized exactly by a rooted-forest criterion and certified by simplicial collapse. That result complements the present theorem: the present paper treats changing carriers, matching and interacting cone defects, and cover-level descent, while the companion order-thinning paper develops the same-carrier incidence theorem and its separate sharpness results.

## 17 Persistence and zigzags

A monotone sequence of inclusions of concept complexes

$$K_0 \hookrightarrow K_1 \hookrightarrow \dots$$

produces an ordinary persistence module on homology. If differentiation alternates between additions, deletions, support restrictions, or reinterpretations, the natural object is instead a zigzag

$$K_0 \rightarrow K_1 \leftarrow K_2 \rightarrow K_3 \leftarrow \dots$$

This is preferable to forcing every cognitive or observational change into a monotone filtration.

Cover-level persistence is already an active theory. In particular, Leitão develops persistent homology directly from cover refinements and proves stability statements that propagate through nerve and co-nerve functors. Therefore any novelty claim here must be made at the level of the specific observation semantics, relative defect interpretation, or descent coupling, not at the level of cover-refinement persistence itself.

## 18 Priority and novelty position

The standard ingredients include Boolean/Hamming and hypersimplex geometry, chain cones, beat-point reductions, relative nerve theorems, sheaf descent, poset-fiber methods, and filtered-complex spectral sequences. For the cover  $X_0 = V(K) \setminus M_0$ ,  $X_1 = V(K) \setminus M_1$ , with  $A = X_0 \cap X_1 = V(L)$ , each  $\Lambda_S$  is precisely the available-vertex obstruction complex  $\text{St}(S, A)$  of Chachólski–Jin–Scolamiero–Tombari [23], with our augmented empty-face convention. Their published Corollary 7.2 treats several outside vertices and Theorem 8.6 controls the homotopy fibers of the vertex-cover inclusion through these obstruction complexes.

The Björner–Wachs–Welker fiber theorem [21] gives decompositions under explicit fiber-connectivity or null-attachment hypotheses. Here the attachment maps are retained and may produce torsion or higher differentials. This is a narrower exact computation, not a replacement for general fiber theory. Likewise, Ferrers graph resolutions, graph independence complexes, and unweighted biclique complexes do not by themselves identify the available-link diagram used here.

The candidate contribution is the explicit observation-product recognition, the coarse-edge-link matching formula, the transitivity restrictions on interacting cores, and the sharp integral positive and negative examples. The matching deletion step is a direct specialization of the obstruction decomposition above; identifying its product-edge link with a suspension of the coarse link is the model-specific calculation. The interaction first page, filtration-width collapse, and two-column kernel/cokernel sequence use standard filtered-chain and mapping-cone algebra after the available-link diagram has been identified. These tools are not claimed as independent new constructions. The historical audit and descent framework provide interpretation rather than priority for classical machinery. No claim of first discovery of the exact specialization is made solely from its absence in a targeted search. The separate convergence report records the primary-source comparison and the remaining publication-level judgment.

## 19 Next theorem target

The matching case, the two-column interaction case, and the degree-two  $E^3$  bound are now established. Unrestricted  $E^2$  collapse in the proposed graph classes is false, so it is no longer an open target.

**Question 19.1** (Integral secondary maps in a four-cycle). *For each integer  $k \geq 2$ , can an exclusive  $K_{2,2}$  profile with torsion-free available-link homology realize  $d^2 : \mathbb{Z} \rightarrow \mathbb{Z}$  as multiplication by  $k$ ? Alternatively, what restrictions does the product-poset geometry impose on the possible integral secondary maps?*

This isolates interaction-created torsion at the smallest exclusive graph that supports a four-core. The unit example above does not resolve it. The smallest possible higher-differential or interaction-torsion realization also remains open globally; induced-restriction minimality is weaker.

## 20 Conclusion

Observation refinement is a transformation between representations, not a chain boundary. Carrier splitting has a neutral-completion factorization, an exact low-height incidence description, and an all-height matching edge-link formula. Beyond matching, available-link inclusions form the interaction complex. Two core columns give an exact chain cone and  $E^2$  collapse; degree two permits a genuine  $d^2$  and only the sharp  $E^3$  bound in general. Interaction can create integral torsion even from torsion-free local links.

Cover refinement and descent answer a separate question on the support actually covered. The historical Boolean reconstruction, local-to-global data, and structural homology therefore remain distinct. The mathematical claims of this draft are the stated specialized formulas and failure boundaries; general topological machinery and publication priority are not silently strengthened.

## A Reproducibility

The accompanying script `verify_unfolding_complexes.py` checks the following finite claims over  $\mathbb{F}_2$  for  $2 \leq m \leq 8$ :

- affine independence of the parent plus all one-flip children;
- exact-layer cardinalities and constant Hamming weights;
- Betti numbers of the cumulative simplex skeletons;
- the relative/mapping-cone rank formula of Theorem 5.2;
- the three-variable  $G, D, E, F$  calculation extracted from the attached logical-complex notes.

The earlier audit scripts separately verify the non-congruence counterexample, the gated Boolean derivative identity, the characteristic-two incidence differential, and the failure of the thesis’s printed gluing identity.

### A.1 Carrier-splitting and interaction verification

The transferred matching gate records 3,242 integral matching cases through four coarse states, 1,707 product-link checks over  $\mathbb{F}_2$ , and 639 additional nonempty matching five-state checks over  $\mathbb{F}_2$ . The interaction gate enumerates 90,201 naturally labelled poset/profile cases through five states over  $\mathbb{F}_2$ , plus 5,000 seeded six-state trials. Coarse-poset isomorphism postprocessing yields 87 poset types and 16,761 profiles; exclusive graphs give 70 colour-preserving types with isolated exclusive vertices retained. These are different counts of the same bounded exhaustive evidence, not additional independent cases.

The main interaction run makes 435 selected integral comparisons, supplemented by nine exact local models, signed integer certificates for both  $d^2$  witnesses, and a separate 32-state torsion construction. The latter checks 181 available links, 31 coarse edge links, and the proved reduced interaction matrix by Smith normal form, not the whole doubled pair. The projective-plane triangulation has facets

012, 013, 024, 035, 045, 125, 134, 145, 234, 235.

An additional 20,000 targeted  $K_{2,2}$  trials locate the sharp secondary differential; these trials are not an exhaustive or uniform isomorphism sample. All coefficients, seeds, scripts, and integer certificates are recorded in the accompanying interaction-gate reports. Computations support the proofs and do not establish global minimality or publication priority.

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