Evolutionary CM/LM Proof Pilot
Compatibility, Closure, and Sharp Robustness Bounds

Research working note prepared for Brian

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Abstract

We distinguish exact Boolean abstraction from uniform separation from the threshold. For a continuous t-norm with an exact half-threshold Boolean abstraction and its standard-negation dual, we prove that the greatest idempotent below an input bound gives the optimal uniform distance from the threshold over all finite formulas. The bound extends to arbitrary synchronous feedback networks in the declared connective language and is sharp already for scalar repeated conjunction. A two-node example has an explicit robustness transition invisible in its constant Boolean trajectory. We also prove a finite-closure obstruction for a polynomial two-node system. The results are a candidate synthesis of classical t-norm and abstraction machinery, not a cleared historical novelty claim.

1 Scope and claim status

This is a separate proof pilot, not a revision of an existing manuscript. The symbolic and numeric CM/LM conventions are inherited from the September 27 foundations manuscript. Its Boolean normal form and pairing theorems are not asserted for graded scalars. The finite-jet comparison uses the October 14 manuscript, not the earlier parent draft.

The main candidate contribution is Theorem 5 and its fragile-alternation example, supported by Theorem 3, its feedback consequence, and the worked coupled example. The exact abstraction criterion, ordinal sums, t-norm powers, and partition refinement have classical antecedents. These elementary proofs establish mathematical statements under explicit assumptions; they do not establish historical firstness. No independent referee or formal proof assistant was used.

2 Compatibility contract

Set $c = 1/2$, $D = [0,1]\backslash\{c\}$, and define $Q(x) = 0$ for $x < c$ and $Q(x) = 1$ for $x > c$. Let $N(x) = 1-x$. A continuous t-norm T is associative, commutative, nondecreasing in each coordinate, and satisfies $T(x,1) = x$. Set $S(x,y) = 1 - T(1-x, 1-y)$. The permitted formula language consists of T, S, N, variables, and constants $0, 1$. In particular, residual implication is not automatically included.

Theorem 1 (Exact cut criterion)**.** *The following are equivalent:*

1. *T maps D^2 into D and $Q(T(x,y)) = Q(x) \wedge Q(y)$ there.*
2. *$T(t,t) > c$ for every $t > c$.*

Either condition implies $T(c,c) = c$. Under these conditions S, N also preserve D and Boolean abstraction, and every finite formula has exact Boolean abstraction.

Proof. Necessity follows by using $x = y = t$. For sufficiency, if an argument is below c, then $T(x, y) \le \min(x, y) < c$. If both are above c, put $t = \min(x, y)$ and use monotonicity. Continuity at (c, c) and $T(c, c) \le c$ imply equality. Standard negation exchanges the two sides of the cut, so duality gives the assertion for S. Structural induction proves the formula statement, including shared variables: no probabilistic independence is assumed. □

For a general cut $c \in (0, 1)$ the same conjunction criterion holds. Standard negation preserves a single complementary cut only at $c = 1/2$. If conjunction compatibility is required at every cut, then $T = \min$: otherwise $T(x, x) < x$ for some x, and any cut strictly between these values contradicts compatibility. Diagonal idempotence implies $T(x, y) = x$ when $x \le y$, by monotonicity and $T \le \min$.

Example 2 (Continuous non-idempotent compatible conjunction)**.** *The ordinal sum of two product t-norms is*

$$T_*(x, y) = \begin{cases} 2xy, & x, y \le 1/2, \\ 1/2 + 2(x - 1/2)(y - 1/2), & x, y \ge 1/2, \\ \min(x, y), & \textit{otherwise.} \end{cases}$$

Boundary formulas agree. This is a continuous t-norm. To see associativity directly, order three arguments. If all are in the same half, affine rescaling reduces to product associativity. If two are in the lower half and one in the upper, both bracketings return the lower-half product. If one is lower and two upper, both return the lower argument. The other axioms follow directly. It satisfies Theorem 1, though $T_*(3/4, 3/4) = 5/8 \ne 3/4$.

The upper-half product can instead be replaced by minimum, leaving the lower-half product intact; call this T_{safe}*. The same proof works. This provides a non-idempotent graded conjunction with strong margin preservation.*

Ordinary product and Lukasiewicz conjunction fail the cut criterion: at $x = y = 3/5$ they give $9/25$ and $1/5$. Minimum passes. Idempotence at the cut alone is insufficient: an upper-half scaled Lukasiewicz component gives $T(3/5, 3/5) = 1/2$.

Residual implication boundary. For T_*, the residuum $x \Rightarrow_T y = \sup\{z : T_*(x, z) \le y\}$ sends $(x, y) = (2/5, 1/5)$ to $1/4$. Both inputs threshold to false, but the result is false, whereas Boolean implication is true. Thus a theorem for the stated De Morgan language cannot be promoted to one for all operations of a residuated fuzzy logic. Material implication $S(N(x), y)$ does satisfy the cut contract.

CM/LM transport. An involutive polarity lift $\mathcal{L}_f(x)[\alpha] = f(x_1^{\langle\alpha_1\rangle}, \ldots, x_n^{\langle\alpha_n\rangle})$, with positive polarity the identity and negative polarity N, retains reindexing equivariance. Pointwise term superposition commutes with this lift and with evaluation in the chosen graded algebra. Crisp coordinate selection also preserves all term operations. After Q, these objects recover the Boolean truth functions, CMs, and symbolic LM lifts in the existing calculus. Graded equality is equality in the declared semantics; Boolean-equivalent formulas need not be graded-equal. For example $T_*(x, x) \ne x$.

The formula frame $\left[\begin{smallmatrix} x & 1-x \\ 1-x & x \end{smallmatrix}\right]$ has determinant $2x - 1$ under real multiplication and is generally not an involution. This refutes direct transplantation of that Boolean proof into real matrix arithmetic; it is not a theorem about every possible fuzzy matrix product.

For real averaging $q_w(a) = \sum_i w_i a_i$, with $w_i \geq 0$, $\sum_i w_i = 1$, preservation of all pointwise products holds iff one weight is one. Indeed, applying multiplicativity to each coordinate indicator gives $w_i = w_i^2$. More generally the product defect is exactly covariance:

$$q_w(ab) - q_w(a)q_w(b) = \mathrm{Cov}_w(a, b).$$

This is classical probability algebra, and explains why soft pairing requires a separate contract.

3 An exact optimal margin theorem

For $a \in (c, 1]$ define the input domain

$$D_a = [0, 1 - a] \cup [a, 1], \qquad e_T(a) = \max\{e \in [c, a] : T(e, e) = e\}.$$

The maximum exists: the set is nonempty by Theorem 1 and closed by continuity.

Theorem 3 (Sharp uniform formula margin)**.** *Assume Theorem 1. Over all finite formulas in $T, S, N, 0, 1$ and all assignments of their variables in D_a,*

$$\inf |\varphi(x) - c| = e_T(a) - c.$$

Every formula output stays in D and has its Boolean formula value after thresholding. If $e_T(a) > c$, every formula output belongs to $D_{e_T(a)}$. The infimum is sharp using only repeated conjunction of one variable set to a.

Proof. Write $e = e_T(a)$. If $e > c$, D_e is closed under the generators. Negation exchanges its two intervals. For conjunction, an argument at most $1 - e$ forces the result to be at most $1 - e$; two arguments at least e give a result at least $T(e, e) = e$. Duality proves closure under S. Constants belong to D_e. Since $D_a \subseteq D_e$, induction gives the lower bound. If $e = c$, the zero lower bound is automatic, and exactness still follows from Theorem 1.

For sharpness, set $p_1 = a$ and $p_{n+1} = T(p_n, a)$. This nonincreasing sequence has a limit L. Every idempotent $d \leq a$ satisfies $p_n \geq d$ by induction. Associativity gives $p_{2n} = T(p_n, p_n)$; continuity yields $L = T(L, L)$. Hence $L \leq e$, while the previous lower bound gives $L \geq e$. Thus $p_n \downarrow e$. Each $p_n > c$ by the exact cut criterion. Therefore $|p_n - c| \downarrow e - c$. □

Corollary 4 (Feedback and the two exactness regimes)**.** *Every synchronous feedback network whose coordinate updates are finite formulas in the stated language has an exact Boolean quotient on D^n. From initial coordinates in D_a, all finite iterates have distance at least $e_T(a) - c$ from the threshold. The bound is optimal across this network class. Exactness permits a zero uniform margin.*

Proof. Coordinatewise application of Theorem 1 gives $Q^n F = F_B Q^n$, and iteration gives $Q^n F^t = F_B^t Q^n$. Invariance of D_e^n gives the positive margin case; the zero case is trivial. For sharpness use the scalar feedback $x_{t+1} = T(x_t, x_t)$ with $x_0 = a$. Its values are the t-norm powers p_{2^t} from the preceding proof and tend to e. □

For T_* the only idempotents are $0, c, 1$. For $c < a < 1$ the optimal uniform margin is zero. By contrast, T_{safe} is idempotent everywhere above c, so its optimal margin is $a - c$. Both have identical Boolean conjunction tables and exact Boolean feedback quotients. Their graded robustness differs sharply.

Interpretation of robustness. The bound concerns distance of exact states and internal gate values from the threshold. Observation errors strictly smaller than that distance preserve Boolean readout. It does not guarantee stability under errors injected back into the dynamics, parameter uncertainty, or perturbations that change the gate law. Such guarantees require a perturbed-system invariant or an error-propagation theorem. Equality at the error bound may hit the undefined threshold.

Finite-depth sharpness. For T_* set $r(x) = 2|x - c|$. Case analysis gives $r(T_*(x, y)) \geq r(x)r(y)$; by duality this holds for S_*, and negation preserves r. On the lower half the expression is $r(x) + r(y) - r(x)r(y)$; on the upper half it is the product; in a mixed pair it is one input margin. A formula with L variable occurrences (counted in its unfolded tree) and input normalized margins at least $m > 0$ therefore has margin at least m^L. Constants have margin one. The bound is attained by an all-conjunction tree on repeated equal true inputs. In particular, self-conjunction feedback satisfies

$$x_t = c + \tfrac{1}{2}(2x_0 - 1)^{2^t}, \qquad x_0 > c.$$

It is Boolean-true at every finite time but tends to the threshold when $x_0 < 1$. An unfolded-tree bound does not claim a circuit-size bound for shared subexpressions.

4 Switching can destroy individually persistent margins

Theorem 5 (Optimal margin for a finite switched library)**.** *Let $T_1, \ldots, T_m$ be continuous t-norms satisfying Theorem 1, with duals S_j and common standard negation. Allow any finite formula using the entire library, and arbitrary switching among such coordinate-update formulas. For initial inputs in D_a, the optimal uniform distance from the threshold is*

$$e_{\mathcal{T}}(a) - c, \qquad e_{\mathcal{T}}(a) = \max\{e \in [c, a] : T_j(e, e) = e \text{ for all } j\}.$$

The lower bound holds at every finite time. Sharpness is achieved as an infimum by a scalar switched self-conjunction trajectory.

Proof. The common idempotent set is closed, contains c, and its greatest element below a makes D_e invariant under every generator. For sharpness let $g(x) = \min_j T_j(x, x)$ and $x_{t+1} = g(x_t)$, $x_0 = a$. The sequence decreases to a limit $L \geq e$. The minimum of finitely many continuous functions is continuous, so $g(L) = L$. Since every $T_j(L, L) \leq L$, equality of their minimum to L implies that every $T_j(L, L) = L$. Hence $L = e$. At each step a minimizing index exists and gives an allowed gate; the resulting finite prefix is a permitted formula or switching schedule. All finite values exceed c. No observability of the schedule is assumed: every selected self-conjunction has the same Boolean update. □

Example 6 (Individually robust laws with fragile alternation)**.** *For $b \in (c, 1)$ let T_b be the ordinal sum of rescaled product t-norms on $[0, c]$, $[c, b]$, and $[b, 1]$, with minimum for pairs in different intervals. On an interval $[l, u]$ its formula is $l + (x - l)(y - l)/(u - l)$. Its idempotents are exactly $0, c, b, 1$, and it preserves the strict half-cut.*

Take $b_1 = 7/10$, $b_2 = 4/5$, and $a = 9/10$. Repeating only T_{b_1} self-conjunction has optimal readout radius $1/5$; repeating only T_{b_2} has radius $3/10$. Alternating them has radius zero, although the Boolean state and the Boolean CM label (AND) stay unchanged at every finite step. The states decrease; both even and odd subsequences have the same limit, which by continuity must be idempotent for both laws. The only common idempotent in $[c, a]$ is c.

This is a concrete failure of separate robustness certificates under composition. Boolean operator labels alone cannot distinguish these semantics. It is the most focused candidate formulation produced by the pilot; its proof combines elementary monotone iteration and classical ordinal sums, so external novelty remains open.

Persistent disturbances require a stronger model. For any t-norm, the perturbed self-conjunction $x_{t+1} = \max(0, T(x_t, x_t) - \eta)$ with fixed $\eta > 0$ decreases by at least η until zero. It leaves the true region in finite time. A universal disturbance guarantee therefore needs restorative dynamics, resets, restrictions on disturbances, or a finite horizon. Readout robustness does not imply this stronger guarantee.

5 A coupled two-node robustness transition

Consider

$$x_{t+1} = T_*(x_t, y_t), \qquad y_{t+1} = S_*(x_t, y_t), \qquad x_0, y_0 \in (c, 1].$$

The Boolean dynamics is $(b, d) \mapsto (b \wedge d, b \vee d)$; in the chosen region its orbit is constantly $(1, 1)$. Put $r_t = 2x_t - 1$, $s_t = 2y_t - 1$, and $k = r_0 + s_0$. Then

$$r_{t+1} = r_t s_t, \qquad s_{t+1} = r_t + s_t - r_t s_t.$$

Theorem 7 (Coupled limit and exact readout radius)**.** *The normalized sum $r_t + s_t = k$ is invariant, and*

$$(r_t, s_t) \longrightarrow (\max(0, k-1), \min(k, 1)).$$

The infimum, over all times and both coordinates, of distance to c equals

$$\frac{\max(0, k-1)}{2} = \max(0, x_0 + y_0 - 3/2).$$

Thus the same constant Boolean orbit admits a positive uniform readout margin precisely when $x_0 + y_0 > 3/2$.

Proof. Conservation is immediate. Since $0 < r_t, s_t \leq 1$, r_t decreases and s_t increases. Their limits satisfy $R = RS$ and $R + S = k$. If $k \leq 1$, positivity of R would give $S = 1$ and $R = k - 1 \leq 0$, so $R = 0$. If $k > 1$, $r_t = k - s_t \geq k - 1 > 0$, so $S = 1$ and $R = k - 1$. To compute the infimum include time zero: $\inf_t r_t = R$, $\inf_t s_t = s_0$, and $R \leq s_0$ because $r_0 \leq 1$. Divide by two. □

For $k < 1$, $r_t \leq r_0 k^t$. For $k = 1$, $r_{t+1} = r_t(1 - r_t)$ and $r_t \sim 1/t$: indeed $1/r_{t+1} - 1/r_t = 1/(1 - r_t) \to 1$, and averaging the increments proves the asymptotic. These rates describe the approach to the threshold, not a finite-time change of truth. For $k > 1$, writing $r_t = k - 1 + u_t$ gives $u_{t+1} = (2 - k)u_t - u_t^2$.

The predicate $x + y > 3/2$ distinguishes persistent readout robustness within a single Boolean state and fixed pair of CM operator labels. It illustrates why margins can be relevant even after exact Boolean closure has been achieved. The recurrence itself is elementary; historical novelty is unestablished.

6 Closure beyond the connective-generated class

Proposition 8 (Exact deterministic quotient)**.** *For a state map $F : Z \to Z$ and observation $P : Z \to A$, a deterministic quotient $G : P(Z) \to P(Z)$ with $PF = GP$ exists iff $P(z) = P(z')$ implies $P(Fz) = P(Fz')$. It is then unique on reachable abstract states.*

Proof. Necessity follows from the commuting equation. For sufficiency define $G(Pz) = P(Fz)$; the hypothesis gives well-definedness. □

This classical fiber criterion applies equally when P includes CM labels, guards, or finitely many memory variables derived from the state. Current operator labels alone are not a closure proof.

Theorem 9 (A polynomial two-node obstruction)**.** *For $F(x, y) = (y^2, x)$ on $(0, 1]^2$, no finite deterministic quotient refining the total observation $B(x, y) = (\mathbf{1}_{x>1/2}, \mathbf{1}_{y>1/2})$ exists. The obstruction persists on the invariant subset whose iterates avoid threshold ties.*

Proof. At even times, $F^{2t}(r, r) = (r^{2^t}, r^{2^t})$. For every integer $j \geq 0$, choose $r_j = 2^{-1/(3\cdot 2^j)}$. Its even-time observations stay $(1, 1)$ for $t \leq j + 1$ and first become $(0, 0)$ at $t = j + 2$. No iterate of either coordinate equals $1/2$, since the relevant exponent is a power of two divided by $3 \cdot 2^j$. Thus these initial states have infinitely many distinct future observation traces. Two states in a single deterministic quotient state would have identical traces, contradicting finiteness. This argument permits arbitrary finite cells, not just polynomial guards. □

The example already fails one-step Boolean closure: $(3/5, 3/5)$ and $(3/5, 9/10)$ have the same observation but different next observations. Adding finitely many instantaneous operator labels cannot repair infinite-horizon deterministic closure. A nondeterministic finite abstraction, or a finite-horizon description, is a different contract and remains possible.

For horizon H, the word $P_H(z) = (Bz, BFz, \ldots, BF^H z)$ is the coarsest observation retaining that entire word. Its next-state map is exact from horizon H to horizon $H - 1$ by deletion of the first symbol. It does not in general give an autonomous fixed-horizon quotient. This is standard finite-trace refinement.

7 Regularity comparison and evolutionary operators

For the multilinear expectation extension $\widetilde{f}$ of a Boolean function, exact half-threshold recovery on all input half-cubes occurs iff f is affine parity. This is a corollary of the local finite-jet rigidity theorem already in the collection: apply it to the polynomial $2\widetilde{f}((u + 1)/2) - 1$ near $u = 0$. Conversely the extension of $f = b \oplus \bigoplus_{i\in I} \alpha_i$ satisfies

$$1 - 2\widetilde{f}(p) = (-1)^b \prod_{i\in I}(1 - 2p_i),$$

which proves threshold compatibility. It is not a new independent theorem claim. The nonsmooth ordinal-sum construction avoids its analytic hypothesis.

For an exogenous sequence of compatible update formulas F_t, stepwise semiconjugacy yields the exact Boolean evolution $b_{t+1} = (F_t)_B(b_t)$. If all steps use the same T, S, N, the idempotent margin certificate remains invariant. Different steps may use different t-norms; Theorem 5 gives the optimal common-barrier guarantee for a finite library under arbitrary switching. If operator choice depends on hidden continuous state, the combined abstraction must satisfy the fiber criterion; the switching sequence cannot silently be treated as known input.

8 Prior art and research decision

1. Klement, Mesiar and Pap, *Triangular norms. Position paper I: Basic analytical and algebraic properties*, https://www.flll.jku.at/div/research/technical_reports/flll-tr-0208.pdf. Indexed Proposition 6.5 and Corollary 6.6 inspected; direct PDF retrieval timed out. These explicitly connect limits of t-norm powers with idempotents and give closedness of the idempotent set. They are direct antecedents to the key ingredient of Theorem 3; the ingredient must not be claimed as new.

2. Fodor, *Left-continuous t-norms in Fuzzy Logic: an Overview*, https://acta.uni-obuda.hu/Fodor_2.pdf. Inspected definitions and structural discussion. Standard t-norm, strong-negation, duality, and ordinal-sum context; no novelty claimed for these.

3. Gispert, Godo and Esteva, *Non-falsity and general threshold-preserving companions of MTL logics*, IJAR 196 (September 2026), 109699, https://doi.org/10.1016/j.ijar.2026.109699. Publisher indexed abstract and excerpts inspected; full theorem-by-theorem comparison remains open. Their consequence relations preserving designated intervals are not automatically the gatewise Boolean homomorphism used here, but this is close priority literature.

4. Petersen et al., *Deep Differentiable Logic Gate Networks*, https://arxiv.org/html/2210.08277. Full relevant real-gate and operator-mixture sections inspected in the preceding assessment. Ordinary product relaxations and learnable gates predate this pilot.

5. Cummins et al., *Boolean models coarsely sample continuous dynamics of regulatory networks*, https://arxiv.org/html/2606.14925v1. Relevant model/refinement sections inspected in the preceding assessment. Discrete/continuous refinement and richer hidden dynamics have strong prior art.

6. *Formal analysis of piecewise affine systems through formula-guided refinement*, https://www.sciencedirect.com/science/article/abs/pii/S0005109812004931. Abstract-level comparison only: formula-relative finite quotients and refinement are established methods.

Targeted searches included continuous t-norm threshold preservation, ordinal-sum Boolean thresholds, t-norm powers and greatest idempotents, coupled product/probabilistic-sum recurrences, and finite deterministic quotients. Search non-detection is not evidence of firstness. The exact powers argument and invariant cut domains are elementary consequences of standard t-norm theory. The manuscript-worthy candidate is the sharp compositional robustness formulation and its evolutionary CM/LM interpretation, not a new fuzzy connective or a new abstraction theory.

Decision. The pilot finds substantive mathematical structure and a sharp theorem with proofs and counterexamples. It does not establish a substantively new result relative to all published mathematics. Preserve this as a technical research note and pursue a focused priority review before developing it into a standalone paper. The finite-library common-barrier theorem is proved here. Stronger remaining targets are constrained switching with feedback disturbances and an explicit observation-cost result; this pilot proves neither. Avoid building a large learner before that decision.

Verification. The companion standard-library script checks exact rational instances of compatibility, associativity, invariant barriers, the coupled recurrence, scalar feedback, residual-implication failure, and finite closure witnesses. Finite checks complement the proofs and do not certify priority. The script writes a JSON receipt beside itself. No original manuscripts are edited.