Signed incidence and a nine-state torsion witness in simultaneous two-bit order thinning

Brian Theory

Technical addendum to *Binary Order Thinning*
Version 1.0 working release candidate — 9 November 2026

Abstract

For a finite poset of height at most two, simultaneous two-bit order thinning has a reduced signed-incidence relative boundary whenever no triangle loses all three edges. Standard signed-graph elimination gives an explicit integral relative homology formula. In this class, integral acyclicity, acyclicity over characteristic two, a rooted-forest condition, and simplicial collapse are equivalent. Acyclicity over a field of characteristic different from two is weaker. We realize the difference with a nine-state poset and an explicit Boolean labeling: the full relative Smith form is $\mathrm{diag}(1^8, 2)$. Both endpoint complexes have circle homotopy type, but their inclusion has degree two on first homology. The example realizes the familiar Möbius-strip boundary obstruction within the specified thinning rule. We make no vertex-minimality or first-discovery claim. Separate interpretation notes explain why structural neutrality does not determine predictive success or sensing cost.

1 Scope and relation to the parent manuscript

The one-bit recognition theorem of [1, Theorem 4.2] is unchanged. Its coefficient independence rests on ordinary oriented incidence. The seven-state integrally neutral noncollapse in [1, Section 7] is also unchanged: it concerns a different obstruction and does not satisfy the local restriction imposed below. Our torsion example is not integrally neutral and does not alter that example's stated minimality result.

The purpose of this addendum is narrow: identify a useful restricted two-bit regime, compute its obstruction groups, and exhibit an actual Boolean realization of the exceptional odd-field case. The signed-graph algebra is classical [2, 3]; rooted-forest and collapse methods have substantial antecedents [4, 5]. In particular, [5, Example 4.9] already exhibits relative $\mathbb{Z}/2$ homology for a triangulated Möbius strip and its boundary. We do not claim that mechanism, Smith-form theory, or the general matching-to-collapse principle as new. The contribution under consideration is the explicit constrained realization and its consequence for the thinning recognition problem. A focused source comparison does not establish external priority.

2 Relative chains and the local restriction

Let P be finite, with height counted by strict inequalities, and let $\alpha = (b_1, b_2) : P \to \{0,1\}^2$. Define a suborder Q on the same carrier by

$$x \le_Q y \quad \Longleftrightarrow \quad x \le_P y \text{ and } \alpha(x) \le \alpha(y) \text{ coordinatewise.}$$

Set $K = \Delta(P)$ and $L = \Delta(Q)$. Orient chains in their poset order. In particular,

$$\partial[xyz] = [yz] - [xz] + [xy]. \tag{1}$$

Assume $\mathrm{ht}(P) \leq 2$. Write E for deleted edges and T for deleted triangles, and set $r = |E|$, $m = |T|$. All vertices survive, so the entire relative chain complex over $\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z}^T \xrightarrow{B} \mathbb{Z}^E \longrightarrow 0. \tag{2}$$

We impose condition **(C)**: no triangle of K has all three edges deleted.

Every deleted triangle has at least one deleted edge. The support $\{xz\}$ is impossible: if xy and yz survive then coordinatewise transitivity retains xz. Thus under (C) the possible nonempty relative boundaries are $+xy$, $+yz$, $+xy-xz$, $-xz+yz$, and $+xy+yz$. The last case requires exactly the label words $(10, 01, 10)$ or $(01, 10, 01)$: both short comparisons fail while the long comparison survives. To check this assertion, if the first label is 00 the first short comparison cannot fail, and if the last is 11 the second cannot fail. The remaining comparable endpoints must be equal to 01 or 10, and the middle must be the other incomparable label. These cases exhaust the Boolean square.

Make one graph vertex for every row $e \in E$. A two-entry column t on rows e, f is an edge with sign

$$\sigma(t) = -B_{e,t}B_{f,t} \in \{1, -1\}. \tag{3}$$

Call this the internal signed graph. In each internal component having singleton columns, adjoin one formal root and join it to the row of each singleton column. Multiple root attachments are allowed. Such a component is *rooted*; otherwise it is *rootless*. A component is *balanced* if every internal cycle has positive sign product. An isolated row is rootless balanced unless it has a singleton attachment. Empty graphs are allowed. The formal roots are algebraic bookkeeping and are not states of P.

3 Integral groups and exact recognition

Lemma 1 (Signed-incidence computation)**.** *Under the hypotheses above, let b and u count rootless balanced and rootless unbalanced components. Then*

$$H_1(K, L; \mathbb{Z}) \cong \mathbb{Z}^b \oplus (\mathbb{Z}/2)^u, \tag{4}$$

$$H_2(K, L; \mathbb{Z}) \cong \mathbb{Z}^{m-r+b}. \tag{5}$$

All other relative integral groups vanish. Over a field k of characteristic two, the dimensions of H_1, H_2 are $b + u, m - r + b + u$; over any other field they are $b, m - r + b$.

Proof. In a connected internal component choose a spanning tree. Each tree column identifies its two row generators up to sign, leaving one generator g. The remaining columns impose no relation on g for positive cycles and impose $2g = 0$ for negative cycles. A singleton column sets $g = 0$ up to sign and kills the component's cokernel. In an unbalanced rootless component the quotient is exactly $\mathbb{Z}/2$, not a further quotient: sending every row generator to $1 \in \mathbb{Z}/2$ satisfies every two-entry relation and is nonzero. This proves (4). The kernel of an integer matrix is free; its rank is $m - \mathrm{rank}_{\mathbb{Q}} B = m - r + b$, proving (5). The same tree elimination over a field gives the stated dimensions, according as two is zero or invertible. Equation (2) proves the remaining vanishings. □

Theorem 2 (Restricted two-bit recognition)**.** *Suppose* $\mathrm{ht}(P) \leq 2$ *and (C) holds. The following are equivalent:*

1. $H_*(K, L; \mathbb{Z}) = 0$*;*
2. $H_*(K, L; \mathbb{F}_2) = 0$*;*
3. $H_*(K, L; k) = 0$ *for every field* k*;*
4. *each augmented component is a tree containing its formal root;*
5. K *simplicially collapses onto* L*;*
6. *the inclusion* $L \hookrightarrow K$ *is a homotopy equivalence.*

The empty relative pair satisfies these statements vacuously.

Proof. Lemma 1 shows that either integral or characteristic-two acyclicity excludes all rootless components. In a rooted component with v row vertices there are at least $v - 1$ internal edges and at least one singleton attachment, hence at least v columns. Vanishing H_2 forces equality in each component: its internal graph is a tree and there is exactly one root attachment. This is precisely (4). Conversely those components have $b = u = 0$ and $m = r$, giving acyclicity over $\mathbb{Z}$ and every field.

In a nonempty rooted tree choose a nonroot leaf. Its row corresponds to a deleted edge occurring in exactly one remaining deleted triangle. No retained simplex can contain that edge, because L is a subcomplex. Height at most two excludes higher cofaces. The edge is therefore a free face, and deleting it with its triangle removes the corresponding leaf and graph edge. Iterate to collapse K onto L. A collapse is a homotopy equivalence, and such an inclusion has zero relative integral homology. These implications close the cycle. □

Proposition 3 (Signs and coefficient choice)**.** *Under the standing hypotheses* $\mathrm{ht}(P) \leq 2$ *and (C),* B *becomes ordinary reduced oriented incidence by row and column sign changes if and only if every internal cycle is balanced. This is also equivalent to total unimodularity. For a fixed field of characteristic different from two, relative acyclicity holds if and only if each augmented component is either a rooted tree or a rootless unbalanced unicyclic graph.*

Proof. Row signs ϵ_e make a two-entry column opposite precisely when $\epsilon_f = \sigma(t)\epsilon_e$. Propagate signs on a spanning tree; extension is consistent exactly when all cycle products are positive. Missing root incidences can be chosen freely, so singleton columns add no switching constraint. Ordinary reduced oriented incidence is totally unimodular. Conversely, an unbalanced simple cycle gives a square submatrix of determinant ± 2: switch its spanning path positive and eliminate its unit entries, leaving a final entry ± 2. This forbids total unimodularity.

For odd characteristic or characteristic zero, Lemma 1 excludes rootless balanced components but not unbalanced ones. Every remaining component has at least as many columns as row vertices. The kernel vanishes exactly at equality, yielding a rooted tree or a rootless connected unbalanced graph with one cycle. □

Thus the torsion in this class is always elementary two-torsion; order four and odd torsion do not occur. None of these signed-incidence calculations by itself proves that a rootless unbalanced component is realizable by Boolean order thinning. The next section supplies that missing realization.

4 A nine-state realization

Take levels $\{0,1\}$, $\{2,3,4,5\}$, $\{6,7,8\}$ and let P be the transitive closure of the covers

$$02, 03, 05, \quad 13, 14, 15, \quad 26, 27, 37, 47, 48, 56, 58. \tag{6}$$

Here ij denotes $i < j$, not a decimal number. Assign

vertex	0	1	2	3	4	5	6	7	8
α	01	10	00	11	00	01	10	00	01

All triangles and their relative boundaries are

026	$02 - 06$	027	$02 - 07$
037	$-07 + 37$	056	$-06 + 56$
058	retained	137	$-17 + 37$
147	$14 - 17$	148	$14 - 18$
156	$15 + 56$	158	$15 - 18$

Thus (C) holds on every triangle, and there are no singleton columns. The retained edges are $03, 05, 08, 13, 16, 26, 27, 47, 48, 58$. Consequently

$$f(K) = (9, 19, 10), \qquad f(L) = (9, 10, 1).$$

Use row order $(02, 06, 07, 14, 15, 17, 18, 37, 56)$ and column order

$$(026, 027, 037, 056, 137, 147, 148, 156, 158).$$

The *full* relative matrix is

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

Theorem 4 (Realized torsion and odd-field failure)**.** *For (6) with the stated labeling, the full Smith form is* $\mathrm{diag}(1^8, 2)$ *and* $\det B = 2$*. Hence* $H_1(K, L; \mathbb{Z}) = \mathbb{Z}/2$ *and all other relative integral groups vanish. Over characteristic two, relative* H_1 *and* H_2 *each have dimension one; over every other field all relative homology vanishes. The inclusion is not a homotopy equivalence and no relative elementary collapse can start.*

Proof. In cyclic order the graph's row vertices are

$$06, 02, 07, 37, 17, 14, 18, 15, 56,$$

with successive columns $026, 027, 037, 137, 147, 148, 158, 156, 056$. All signs are positive except that of 156, whose two coefficients are $+1$. Removing 156 leaves an ordinary oriented spanning tree; its

eight unit relations identify all row generators with a single g. Restoring 156 imposes exactly $2g = 0$. Mapping every row generator to $1 \in \mathbb{Z}/2$ proves $g \neq 0$. Tree elimination gives eight Smith entries equal to one, and the remaining relation gives two. Exact rational elimination in the recorded ordering gives determinant $+2$; the sign is immaterial to the group. The field statements follow from this full Smith form, not from testing finitely many primes. Integral relative homology forbids homotopy equivalence. Every deleted edge belongs to exactly two deleted triangles, excluding a first free pair. □

The distinction between a full matrix and a minor matters: a determinant-two minor proves failure of total unimodularity but not torsion of the full cokernel. Extra singleton relations can kill that torsion. There are no such extra columns in (7).

5 Absolute topology and the map

The following edge/triangle collapses of K are valid in order:

$$03/037,\ 07/027,\ 02/026,\ 06/056,\ 05/058,$$
$$13/137,\ 16/156,\ 15/158,\ 17/147,\ 14/148.$$

The resulting graph has nine vertices and nine edges, is connected, and has cycle $2{-}6{-}5{-}8{-}4{-}7{-}2$ and three pendant edges. In L, collapse 05/058 and then the vertex/edge pair 5/58. What remains is the cycle

$$z = 0 - 3 - 1 - 6 - 2 - 7 - 4 - 8 - 0. \tag{8}$$

Thus $K \simeq S^1 \simeq L$, and for either endpoint the integral groups are $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}$, and $H_i = 0$ for $i \geq 2$. The long exact sequence gives

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Consequently i_* is multiplication by ± 2. Choosing generator signs makes it $+2$. The certificates independently compute -2 with their fixed orientations: take the primitive cocycle dual to the sole chord of a spanning tree of the collapsed K graph, extend it back across the elementary collapses, and evaluate it on (8).

The geometric interpretation is also checkable. All vertex links of K are paths except the link of vertex 5, which is a circle. Its boundary edges form exactly (8). It is a connected compact surface with one boundary component and Euler characteristic zero, hence a Möbius strip by the classification of compact surfaces. The subcomplex L consists of the boundary plus the filled triangle 058. This interpretation is explanatory; the algebraic and collapse proofs do not depend on recognizing the surface from a picture.

6 Limits and operational interpretation

Height. At greater height, $\ker B$ need not be H_2: deleted tetrahedra supply d_3. For the four-chain $0 < 1 < 2 < 3$ with one active bit 0100, the relative matrices are

$$d_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad d_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

The graph for d_2 has a cycle, yet the pair collapses by 123/0123, 12/012, 13/013. Theorem 2 must not be used by ignoring higher differentials.

Prediction and execution. Homotopy neutrality is not an information-gain score. For noiseless observations with fibers B_s, prior μ, and target c under zero–one loss, elementary fiberwise optimization gives

$$\text{optimal success} = \sum_s \max_y \mu(B_s \cap c^{-1}(y)).$$

Under a uniform identity task on n states this equals the number of nonempty fibers divided by n. For the singleton indicators of states $1, \dots, n-1$ with anchor 0, all signature posets are stars and all forgetting maps are homotopy equivalences, while success increases from $1/n$ to one. On the background order $0 < \{1, \dots, n-1\}$ all those tests are isotone and hence legal on every support. This is consistent with, and sharpens, the distinction already made in [6, Sections 11.1 and 15].

The actual observation fibers and their probabilities matter; their homology does not determine them. Costed acquisition additionally requires the supplied sensor presentation. For identity recovery on four equiprobable states with an antichain background, the two coordinate tests and their complements, and the four singleton indicators, each generate the discrete positive topology. Their optimal expected exact costs under unit test costs are respectively 2 and $9/4$, and worst-case costs 2 and 3: singleton tests necessarily have leaf depths $1, 2, 3, 3$, whereas two coordinate readings suffice. Any binary decision tree identifying four equiprobable states has expected depth at least two, so the coordinate cost is optimal. These are finite operational examples, not a new general decision-theoretic theorem.

Execution order is a separate issue. For $P = \{a\} < \{b\} < \{c, d\}$ with labels $(10, 01, 10, 11)$, the simultaneous relative matrix, in rows (ab, bc) and columns (abc, abd), is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The block collapses by bc/abc, then ab/abd. But the first bit alone deletes ab and both triangles, giving matrix $(1\ 1)$ and relative $H_2 = \mathbb{Z}$; that first step is illegal. The second bit first deletes only bc/abc, after which the first bit deletes ab/abd, both legal. Under support-conditioned noiseless execution, the second bit instead leaves $\{a, c\}$ or $\{b, d\}$, on which the first is respectively constant or ascending. These are distinct protocol calculations. No claim that every simultaneous legal block has a legal sequential ordering is made.

The nine-state witness itself is an obstruction, not a legal sensing protocol. Each physical bit alone has relative $H_1 = 0$ and $H_2 = \mathbb{Z}$, as verified by a full-row unit minor of its 5×6 boundary. The simultaneous block is also illegal under a homotopy-equivalence gate. Neither formal admissibility nor the word "safe" establishes physical safety [7]. No consciousness, empirical validation, or architecture-superiority claim follows.

Unresolved questions. Nine is a certified upper bound, not a proved minimum. No exhaustive minimum-size claim is part of this addendum. Disjoint unions realize $(\mathbb{Z}/2)^k$ in relative H_1 on $9k$ states, but this is not a connected minimal-size assertion. The exclusive $K_{2,2}$ secondary-map Question 19.1 in [6] has different geometric constraints and is not resolved by this witness.

7 Verification and release status

The accompanying standard-library scripts reconstruct comparabilities, order complexes and every relative column directly from the covers and labels. Two implementations provide full integer matrices U, V satisfying $UBV = \operatorname{diag}(1^8, 2)$, with $\det U, \det V \in \{1, -1\}$. They verify the absolute collapse sequences, vertex links, boundary cycle and primitive-cocycle pairing, as well as the individual-bit unit minors. Modular ranks at selected primes are corroboration only; the Smith certificate proves the statement over all fields.

The package separates analytic proofs, exact fixed-example computation and literature positioning. It includes a summary of specialist-style and hostile AI reviews of supplied draft text, with substantive responses. The reviewers did not independently execute the final code or inspect the PDF; the authoring agent performed those checks separately. These are AI reviews, not external human peer review. No website change, public deposit or external review request is made by this local preparation. This addendum is intended to accompany the parent paper, rather than to inflate standard signed-graph algebra into a standalone priority claim.

References

[1] B. Theory. *Binary Order Thinning of Finite Posets: Homology, Collapses, and Sharp Limits*. 27 September 2026. https://www.b-theory.com/manuscripts-html/binary-order-thinning/.

[2] T. Zaslavsky. Signed graphs. *Discrete Applied Mathematics* 4 (1982), 47–74. doi:10.1016/0166-218X(82)90033-6.

[3] T. Zaslavsky. *Matrices in the Theory of Signed Simple Graphs*. arXiv:1303.3083v1, Sections IV.B–IV.C. https://arxiv.org/abs/1303.3083.

[4] O. Bernardi and C. J. Klivans. *Directed rooted forests in higher dimension*. arXiv:1512.07757v4. https://arxiv.org/abs/1512.07757.

[5] I. Contreras and A. R. Tawfeek. *On discrete gradient vector fields and Laplacians of simplicial complexes*. arXiv:2105.05388v2, Theorem 4.2 and Example 4.9. https://arxiv.org/html/2105.05388v2.

[6] B. Theory. *Successive Concept Differentiation as Observation Refinement: Carrier Splitting, Interaction-Core Homology, and Descent*. Preserved v0.5, 29 September 2026. https://www.b-theory.com/manuscripts-html/observation-refinement/.

[7] B. Theory. *From Observable States to Acquirable Predictive Models*. Revised 7 October 2026. https://www.b-theory.com/manuscripts-html/acquirable-predictive-models/.