Protective Switching and Sharp Observation Margins in a Two-Node Ordinal-Sum System

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Abstract

For fixed $0 < \alpha < \beta < 1$ we analyze a two-node product-ordinal-sum network, with $h = 1-\beta$ independent of α. The range $\beta/2 \le \alpha < \beta$ is sufficient for global zero-basin identification, first-use protection at every beta-critical point, and exact periodic worst schedules under a recurring gap constraint. The sharp leading margins are $(2\beta/h)n^{-3}$ and $(2\beta/h)\zeta(3)K^{-3}$. Parameter-dependent energy estimates prove the coefficients. A certified critical-input counterexample outside this range has zero margin even under alternating switches. The sufficient range is not claimed maximal. Late forced first-use asymptotics remain valid for every fixed strict parameter pair, despite possible failure of early protection and recurring robustness.

Keywords: ordinal sums; monotone maps; constrained switching; critical basins; observation margins.

1 Introduction

Graded realizations of Boolean readouts distinguish a positive logical output from its distance to the threshold. We study how switching changes that distance in a concrete two-node network. The principal questions are global: which initial states admit zero margin, whether an eventual protective mode is sufficient, and which admissible schedule gives the smallest margin. The answers involve an exact basin boundary, a realizable greedy envelope, and an exchange inequality identifying a periodic worst word.

This system supplies one explicit dynamical realization relevant to evolutionary correspondence-matrix/logical-matrix (CM/LM) representations: the logical conjunction/disjunction readouts are retained while a mode selects the graded operation. The representation alone does not determine this dynamics; the state domain, ordinal sums, duality and switching contracts below are additional declared choices. Finite compositions remain maps of the invariant state domain, but are not asserted to remain in the two-element family of one-step maps. No closure under a new matrix product or inverse is needed or claimed. The mathematical results concern this specified realization, rather than every evolutionary CM/LM system.

The local critical geometry belongs to established nilpotent-parabolic theory [1]. Our proofs are elementary and self-contained; the model-specific schedule comparison, terminal-margin transfer and certified obstruction are the contribution examined here. Related constrained-switching and monotone safety results [2, 3] are distinguished in the final discussion. Historical priority remains unresolved.

2 Maps and contracts

Fix $0 < \alpha < \beta < 1$, and set $h = 1 - \beta > 0$. Throughout, h is not identified with α. On $(0,1]^2$ define

$$F_q(r,s) = (U_q(r,s), V(r,s)), \quad V(r,s) = r + s - rs, \quad q \in \{\alpha, \beta\},$$

$$U_q(r,s) = \begin{cases} rs/q, & r, s \le q, \\ q + (r-q)(s-q)/(1-q), & r, s \ge q, \\ \min(r,s), & \text{otherwise.} \end{cases}$$

For completeness, let $b_q = (1+q)/2$ and define T_q on $[0,1]^2$ as the product ordinal sum with blocks $[0,1/2], [1/2, b_q], [b_q, 1]$. Within a block $[l,u]$, its value is $l + (x-l)(y-l)/(u-l)$; between blocks it is $\min(x,y)$. Set $S_q(x,y) = 1 - T_q(1-x, 1-y)$. The original update is $(x', y') = (T_q(x,y), S_q(x,y))$ on $(1/2,1]^2$. The normalization $r = 2x - 1$, $s = 2y - 1$ gives the displayed maps: complementary inputs lie in the common lowest block, hence V is independent of the mode. All finite states stay strictly above the threshold $1/2$. An observation error strictly smaller than the margin in each coordinate preserves this positive threshold readout. No error is injected into the state recurrence in this definition. A schedule is an infinite word on $\{\alpha, \beta\}$; adaptive choices also produce such a word along each trajectory. It is called fair if each symbol occurs infinitely often, with no bound on gaps. The full-trajectory observation margin is $\frac{1}{2} \inf_{t \ge 0} \min(r_t, s_t)$. All schedule infima fix the initial state. Each mode is isotone, r decreases, s increases, and $r_1 \le \min(r_0, s_0)$. Limits obey $R(1-S) = 0$.

At a beta-critical point defined below, let $z_n = F_\beta^n z_0$. M_n is the infimum of that margin over words permitting exactly n beta updates, then one alpha, then an arbitrary tail; the prefix and time zero are included. $D_L = \min_{0 \le n \le L} M_n$ is the first-use deadline value. G_K, for $K \ge 2$, is the infimum over words with no K consecutive betas, including in the initial block; thus the first alpha index is at most $K-1$. Let P_K denote the full-path margin of the particular word $p_K = (\beta^{K-1}\alpha)^\infty$, whether or not it is optimal.

3 The beta-critical boundary for arbitrary beta

Lemma 1. *For every $0 < \beta < 1$, the constant-beta zero-margin basin is $0 < s < \beta$, $0 < r \le \phi_\beta(s)$, where the boundary function is unique, continuous, strictly decreasing and involutive. Its endpoint limits are $\phi_\beta(0+) = \beta$ and $\phi_\beta(\beta-) = 0$. Boundary orbits tend to $(0, \beta)$; strictly lower orbits tend to $(0, S)$, $S < \beta$. Write $d = \beta - s$. Then*

$$\mathcal{A}_\beta(s) = \frac{d^2}{2\beta(h+d)} < \phi_\beta(s) < \mathcal{B}_\beta(s) = \frac{d^2}{\beta(h+d)} < d.$$

A beta-critical state means precisely a point on this boundary.

Proof. For the beta law, reaching $s \ge \beta$ preserves a positive floor $\min(r, \beta)$. Conversely $R > 0$ forces $S = 1$, hence a finite crossing. Also $U_\beta \ge rs$, strictly in the open square, so $r + s$ strictly increases there. Every zero orbit thus stays in $r + s < \beta$ and satisfies

$$r' = r(1 - d/\beta), \qquad d' = d - (h+d)r. \tag{1}$$

The region $r \le \mathcal{A}_\beta(s)$ is invariant: on its boundary, $d' = d - d^2/(2\beta) > 0$ and direct subtraction yields

$$\mathcal{A}_\beta(s') - r' = \frac{d^4(1 + \beta - d)}{8\beta^3(h+d)(h + d - d^2/(2\beta))} > 0.$$

For smaller r, both successor coordinates decrease and $\mathcal{A}_\beta$ decreases with s, proving invariance. If $S < \beta$, then $r_t \to 0$ and $\mathcal{A}_\beta(s_t) \to \mathcal{A}_\beta(S) > 0$, giving eventual strict entry.

The positive basin is open by finite crossing (an equality hit is followed by a strict crossing); the zero basin is relatively closed and downward closed. Each vertical slice has an attained positive zero endpoint below $\beta - s$. Its limit must have $S = \beta$, since otherwise strict certificate entry and continuity would give larger zero points on that slice. Distinct ordered zero orbits have strictly ordered terminal second coordinates: after at most one step their second-coordinate gap is positive and is bounded below subsequently by multiplication by $1-\bar{r}_t$. The product equals $(1-\bar{S})/(1-\bar{s}_m) > 0$. Hence a smaller point has $S < \beta$, and every strict coordinatewise increase of a boundary point has positive margin. This proves strict decrease. Points strictly below the graph lie in the open zero interior by eventual strict certificate entry; points above lie in the open positive basin. These two neighborhoods prove continuity of the graph. Input symmetry gives identical successors after swapping coordinates, so the graph is symmetric and involutive. Its endpoint limits follow from $\phi_\beta(s) < \beta - s$ and involution.

The strict inner bound follows from strict certificate entry. For any zero orbit, set $g(s) = (\beta - s)/[\beta(1 - s)]$, which strictly decreases. Then $r_t - r_{t+1} = g(s_t)(s_{t+1} - s_t)$. Summing and using positive later increments gives $r_0 < g(s_0)(\beta - s_0) = \mathcal{B}_\beta(s_0) < \beta - s_0$. □

4 A sufficient parameter range and exact schedule ordering

Let $H = (\min(U_\alpha, U_\beta), V)$. Isotonicity and the shared second coordinate imply that one mode realizes H at each step and $H^t(z)$ lies below every switched trajectory. Induction uses $H(z_t^H) \le F_q(z_t^H) \le F_q(z_t^q)$. Thus the sharp arbitrary-word margin is the greedy terminal first coordinate divided by two. Every finite greedy prefix has a fair extension, so arbitrary/fair uniform infima agree; this does not assert fair attainment.

Theorem 2 (Global sufficient range)**.** *If $\beta/2 \le \alpha < \beta$, the switched zero basin equals the constant-beta zero basin. Every first alpha use on a beta-critical orbit yields positive margin for all later switching, including the full prefix. These statements include $\alpha = \beta/2$. The range is sufficient; maximality is not claimed.*

Proof. For either mode, $s \ge \beta$ preserves $r \ge \min(r_{\text{current}}, \alpha) > 0$. Therefore a zero greedy orbit has terminal $S \le \beta$. Since $U_q > rs$ in the open square, all its finite states satisfy $r + s < \beta$. In this triangle $\min(r, s) < \beta/2 \le \alpha$ and $\max(r, s) < \beta$. Writing $m = \min(r, s)$, $M = \max(r, s)$, one has $U_\alpha = m \min(M/\alpha, 1) > mM/\beta = U_\beta$. Thus every zero greedy orbit is a beta orbit. Conversely a zero beta orbit bounds the greedy orbit above, forcing zero greedy margin. At any critical point, the alpha successor has the same second coordinate and a strictly larger first coordinate than the critical beta successor. The boundary separation proved above places it in the positive switched basin. All prefix coordinates are positive, so the complete trajectory has positive margin. The strict inequality $r + s < \beta$ removes any endpoint ambiguity when $2\alpha = \beta$. □

Proposition 3 (Fairness and uniform margin)**.** *For any strict parameter pair, the uniform infima over arbitrary and fair words agree and equal $R_H/2$, where R_H is the terminal first coordinate of the greedy orbit. This value is positive exactly when that orbit reaches $s \ge \beta$ at a finite time. If $\beta/2 \le \alpha < \beta$, every critical initial point has positive individual margin under each fair word, but its uniform fair infimum is zero. Below the critical graph a fair zero-margin word exists. Above it all words share a positive uniform margin.*

Proof. The envelope comparison gives the lower bound and an arbitrary attaining word. Every finite greedy prefix admits a fair extension; taking increasingly long prefixes proves equality of the uniform infima. Since $r_1 \le s_0$, the full-path infimum is the terminal first coordinate divided by two. After $s \ge \beta$, either $r \le \alpha$ and both laws fix r, or $r \ge \alpha$ and both preserve the floor α. Conversely $R_H > 0$ forces $S_H = 1$ and therefore a finite crossing.

In the sufficient range, Theorem 2 applies at the first alpha of any fair word. Its finite preceding prefix has positive coordinates. The constant-beta critical orbit gives zero arbitrary, hence zero uniform fair margin. Below the graph the beta orbit tends to $(0, S)$ with $S < \beta$. Choose $S < c < \beta$ and a sufficiently late prefix so that $r < c$, $s < c$, and $2r/(1 - c/\beta) < c - s$. Alternate beta-alpha from there. Before any putative first exit above c, a pair contracts r by at most c/β and raises s by at most twice its starting r. Summing the geometric budget excludes both a whole-pair and a half-pair exit. Thus $s < c$ forever and $r \to 0$, giving a fair zero word. The above-graph assertion follows from global basin equality. No fair attainment of a positive worst margin is asserted by equality of infima alone. □

Lemma 4 (Exchange for every strict pair)**.** *For any $0 < \alpha < \beta < 1$, the region $\Omega_\alpha = \{\min(r, s) \le \alpha\}$ is invariant and*

$$F_\beta \le F_\alpha, \qquad F_\alpha \circ F_\beta \le F_\beta \circ F_\alpha \quad \text{on } \Omega_\alpha.$$

The word p_K minimizes both coordinates at every finite horizon over words with no K consecutive betas, for any start in this region.

Proof. Set $f_q(u) = \min(u/q, 1)$. On this region $U_q = m f_q(M)$; after one update $r \le m \le \alpha$, proving invariance. The ratio f_β/f_α equals α/β up to α, then u/β up to β, then 1; it is nondecreasing. With $v = V(r, s) \ge M$, this orders $m f_\beta(M) f_\alpha(v) \le m f_\alpha(M) f_\beta(v)$, the two first coordinates of the compositions. Their second coordinates are ordered by $f_\beta \le f_\alpha$ and isotonicity of V.

For a finite word, match the periodic prefix left to right. At the first mismatch the target symbol must be beta: a target alpha is forced by the preceding $K - 1$ matched betas. Move the first later beta left across the intervening alphas, or replace the current alpha if no beta remains. Each swap alpha-beta to beta-alpha lowers the terminal state by the exchange and isotonic suffix maps. It is legal: preceding beta-run length is zero except at the final swap, where the matched prefix has length less than $K - 1$; the following beta-run shortens. Replacement with an all-alpha suffix is also legal and lowers the state. Induction gives the periodic prefix. Every legal finite word has an infinite legal extension, completing the comparison. □

For a critical state already in Ω_α, the beta law is the greedy law for all future time, even when $\alpha < \beta/2$. A first alpha strictly raises the state above the critical beta successor, so the constant-beta basin theorem and the same envelope comparison give positive tail margin. Hence

$$G_K = P_K > 0 \quad \text{for every } K \ge 2 \tag{2}$$

for these state-specific starts. In the sufficient parameter range every critical point belongs to Ω_α, since its minimum is below $\beta/2$. Positive margin forces a finite $s \ge \beta$ crossing; after the first update $r \le \alpha$ and both modes freeze r at crossing. The terminal R controls the whole path because $R \le r_1 \le s_0$, so its original margin is $R/2$. Requiring both symbols in every same-length K block, or adding ordinary fairness, leaves this sharp value unchanged because p_K lies in each class.

5 Parameter-dependent almost-invariant bounds

Use the following constants; none assumes $\alpha = h$:

$$C = \frac{1}{2\beta h}, \quad A = \frac{1}{2\beta}, \quad B = \frac{2\beta + h}{6\beta^2 h^2}, \quad L = \frac{1}{\beta h}, \quad D = \frac{3\beta + h}{2\beta^2 h}.$$

Choose

$$\delta = \min\left\{\frac{\beta}{4}, \frac{h}{2}, \frac{\beta - \alpha}{2}, \sqrt{\frac{\alpha\beta h}{2}}\right\} > 0, \quad J = (h + \delta)L, \quad S = 2/h.$$

Then $L\delta^2 \le \alpha/2$, $J\delta \le 3/8$, $A\delta \le 1/8$. Define $E(r, d) = r - Cd^2 - Ard + Bd^3$. Direct expansion of (1) gives

$$\Delta_\beta E = -3Brd^3 + Ar^2 d + Dr^2 d^2 + 3Br^2 d^3 - B(h + d)^3 r^3. \tag{3}$$

Set

$$c_1 = 3B, \quad c_2 = A + D\delta + 3B\delta^2, \quad c_3 = B(h + \delta)^3, \quad Q = S(c_1 + c_2 L + c_3 L^2 \delta),$$
$$X = CJ^2 + ALJ + BJ^3\delta^2, \quad H_1 = 2Q + X + AL/\beta, \quad H_* = 2Q + X + 3AL/\beta.$$

These are finite positive fixed-parameter constants, not uniform constants over all parameter pairs.

Lemma 5. *On any segment whose prestates satisfy $0 \le d_t \le \rho \le \delta$, $r_t \le L\rho^2$ and $\sum r_t \le S\rho$, the sum of absolute hypothetical beta energy drifts is at most $Q\rho^4$. At a critical point with $d \le \delta$,*

$$|E(r, d)| \le Qd^4, \qquad r = Cd^2 + (AC - B)d^3 + O(d^4), \quad AC - B = \frac{h - 4\beta}{12\beta^2 h^2}.$$

Consequently $nd_n \to 2\beta$ and $n^2 r_n \to 2\beta/h$ on every fixed critical orbit. The sign of the cubic coefficient is not assumed negative.

Proof. Equation (3) bounds the absolute drift by $c_1 rd^3 + c_2 r^2 d + c_3 r^3$. Summing and using $r \le L\rho^2$, $d \le \rho \le \delta$ gives the stated $Q\rho^4$. On a critical tail, telescoping (1) gives

$$r = \sum_{t \ge 0} \frac{d_t}{\beta(h + d_t)}(d_t - d_{t+1}) \le \frac{d^2}{\beta(h + d)} \le Ld^2, \qquad \sum r_t \le d/h \le Sd.$$

Since E tends to zero, the drift estimate proves the critical energy bound. Solving $r(1 - Ad) = Cd^2 - Bd^3 + E$ gives the expansion, with remainder bounded by $[Q + A|AC - B|]d^4/(1 - A\delta)$. In particular $r_n \sim Cd_n^2$ and

$$\frac{1}{d_{n+1}} - \frac{1}{d_n} = \frac{(h + d_n)r_n}{d_n d_{n+1}} \to hC = 1/(2\beta).$$

Cesaro summation gives the two orbit limits. □

6 One-use and recurring transfer

Take a critical pre-alpha state with $d = \rho \le \delta$. Its coordinates obey $r \le L\rho^2 < \alpha$ and $s \ge \beta - \delta > \alpha$. Before escape alpha leaves r fixed; beta decreases it by $\epsilon = rd/\beta$. Both modes have $d' = d - (h + d)r$. Alpha energy change is exactly

$$\Delta_\alpha E = \Delta_\beta E + \epsilon(1 - Ad'). \tag{4}$$

The critical first beta successor has $d' > 0$ since $d' \geq \rho - \rho^2/\beta > 0$. The alpha successor strictly exceeds it in r and belongs to Ω_α, so its arbitrary tail has positive uniform margin by Lemma 1 and the envelope argument. Thus a crossing occurs in finite time under the beta-greedy or periodic tail.

At crossing, $-J\rho^2 \leq d_\tau \leq 0$, $R \leq L\rho^2$, and $\sum_{t<\tau} r_t \leq (\rho + J\rho^2)/h < S\rho$. Both laws freeze r thereafter. The exit error is $|E(R, d_\tau) - R| \leq X\rho^4$. Initially $|E| \leq Q\rho^4$; all hypothetical beta drifts together cost at most $Q\rho^4$, including the last update whose new d' can be negative. Also $|d'| \leq \rho$. For one alpha followed by beta-greedy switching, (4) therefore gives

$$|2M_n - r_n d_n/\beta| \leq H_1 d_n^4 \qquad (d_n \leq \delta). \tag{5}$$

Here the entire initial beta prefix is included: terminal $R \leq r_n \leq s_0$ for $n \geq 1$; for $n = 0$, the alpha conjunction itself is at most s_0. All earlier first coordinates are at least R and second coordinates at least s_0. Hence the sharp full-path margin equals $R/2$.

For the periodic word set $t_j = jK - 1$ and take $\rho = d_{K-1}$ before its first alpha, and set $r_* = r(t_1) = r_{K-1}$. Write $\epsilon_{K,j} = r(t_j)d(t_j)/\beta$ before crossing and zero after. Let $S_K = \sum_j \epsilon_{K,j}$. Each nonzero kick after the J_0th is preceded by $K - 1$ beta losses at least as large as that kick, since positive r, d decrease. The blocks are disjoint, so

$$\sum_{j>J_0} \epsilon_{K,j} \leq \frac{r(t_{J_0})}{K-1} \tag{6}$$

when t_{J_0} precedes crossing, and the tail is zero otherwise. In particular $S_K \leq r_*\rho/\beta + r_*/(K-1)$. For $K\rho \geq \beta$, $1/(K-1) \leq 2\rho/\beta$, giving $S_K \leq 3L\rho^3/\beta$. The total error in dropping the factors $1 - Ad'$ is at most $A\rho S_K$. Thus the periodic terminal coordinate satisfies

$$|2P_K - S_K| \leq H_*\rho^4 \quad (\rho \leq \delta,\ K\rho \geq \beta). \tag{7}$$

Both conditions hold eventually for every fixed parameter pair and critical start since $Kd_{K-1} \to 2\beta$.

Theorem 6 (Leading laws and their scope)**.** *For every fixed $0 < \alpha < \beta < 1$ and fixed beta-critical initial point,*

$$M_n \sim \frac{2\beta}{h} n^{-3}, \qquad P_K \sim \frac{2\beta}{h}\zeta(3)K^{-3}.$$

If the critical start lies in Ω_α, then $G_K = P_K$ for all $K \geq 2$, so the latter law holds for G_K. In particular it holds for every critical start throughout the sufficient range $\beta/2 \leq \alpha < \beta$. In that range $D_{K-1} \sim (2\beta/h)K^{-3}$ and $G_K/D_{K-1} \to \zeta(3)$. Early M_n or D_L outside that range need not be positive; no recurring-law assertion is made there without a state condition.

Proof. From (5) and the critical limits, $n^3 M_n \to \frac{1}{2\beta}(2\beta/h)(2\beta) = 2\beta/h$. This is local along the late beta orbit and does not require the sufficient global range. For each fixed integer j, scaled periodic pre-kick coordinates obey $K^2 r(t_j) \to (2\beta/h)j^{-2}$ and $Kd(t_j) \to 2\beta/j$. To justify this, start at $t_1 = K - 1$, set $u = K^2 r$, $v = Kd$, $\theta = 1 + (t - t_1)/K$. Beta updates are $u^+ = u - uv/(\beta K)$, $v^+ = v - hu/K - uv/K^2$; alpha changes only the first formula to $u^+ = u$. On $1 \leq \theta \leq j$ before and including a possible first crossing, scaled prestates are bounded. The limiting equations $\dot{u} = -uv/\beta$, $\dot{v} = -hu$ have solution $(u, v) = ((2\beta/h)\theta^{-2}, 2\beta/\theta)$. On a compact rectangle the vector field is Lipschitz. Taylor comparison and the at most j alpha omissions yield $e_{m+1} \leq (1 + L_j/K)e_m + C_j/K^2 + (C_j/K)\mathbf{1}_\alpha$. Iteration gives $\max e_m \leq e^{L_j j}(e_0 + C'_j/K) = o(1)$. A possible first crossing would contradict the strictly positive limiting $v \geq 2\beta/j$, closing the stopped estimate. Constants here fix j.

Thus $K^3\epsilon_{K,j} \to (4\beta/h)j^{-3}$ for each fixed j. By (6), the scaled tail beyond J_0 has limsup at most $(2\beta/h)J_0^{-2}$. Truncate, take $K \to \infty$, then $J_0 \to \infty$ to obtain $K^3 S_K \to (4\beta/h)\zeta(3)$. The error in (7) vanishes after multiplication by K^3, proving the law for P_K. Equation (2) gives the G_K claim. In the sufficient range all finitely many early M_n are positive. The asymptotic bounds on later M_n and that positive finite minimum imply $D_L \sim (2\beta/h)L^{-3}$, without assuming monotonicity. □

At $\beta = 3/5$ the coefficients reduce to 3 and $3\zeta(3)$.

7 A fixed-parameter logarithmic refinement

Proposition 7. *Only in this proposition, fix $(\alpha, \beta) = (2/5, 3/5)$ and a fixed beta-critical initial point. Then*

$$M_n = \frac{3}{n^3} - \frac{21 \log n}{n^4} + O(n^{-4}).$$

The remainder constant and onset may depend on the initial point.

Proof. Lemma 5 gives $r = (25/12)d^2 - (625/216)d^3 + O(d^4)$. Substitution into $d' = d - (2/5 + d)r$ yields $d' = d - (5/6)d^2 - (25/27)d^3 + O(d^4)$. Consequently, for $w_n = 1/d_n$,

$$w_{n+1} - w_n = a + \frac{b}{w_n} + O(w_n^{-2}), \qquad a = 5/6, \quad b = 175/108.$$

Already $w_n \sim an$ by Lemma 5. First summation gives $w_n = an + O(\log n)$. Hence $1/w_n - 1/(an) = O(\log n/n^2)$ is absolutely summable, as are the $O(w_n^{-2})$ terms. Summing once more gives

$$w_n = \frac{5}{6}n + \frac{35}{18}\log n + O(1), \qquad d_n = \frac{6}{5n} - \frac{14 \log n}{5n^2} + O(n^{-2}).$$

The one-use transfer bound (5) implies $M_n = (125/72)d_n^3 + O(d_n^4)$. Expanding its cubic proves the stated coefficients 3 and -21. All estimates use fixed parameters and a fixed critical orbit; no parameter-uniform logarithmic law is asserted. □

8 A certified failure outside the sufficient range

Take $\beta = 3/5$, $\alpha = 1/100$, $s_0 = 1/5$, and the exact implicit critical coordinate $r_0 = \phi_\beta(1/5)$. The rational certificates detailed below establish

$$11/50 < r_0 < 6/25. \tag{8}$$

The lower endpoint enters the strict beta zero certificate after three exact updates, with slack $666485001263/272735592187500 > 0$. The upper endpoint crosses $s = \beta$ by update 23, certified by downward rounding its orbit, so is beta-positive. Lemma 1 yields the strict bracket, rather than assigning exact criticality to a rounded point.

Throughout this initial interval both inputs exceed alpha, and

$$U_\beta(r, 1/5) - U_\alpha(r, 1/5) = \frac{2(35r - 2)}{495} > 0.$$

Thus the first alpha successor lies strictly BELOW the critical beta successor with the same second coordinate. Lemma 1 implies its constant-beta tail has zero margin; hence $M_0 = 0$. This is a long-time basin conclusion, not merely the one-step comparison.

More strongly, $(\alpha\beta)^\infty$ has zero margin from every initial point in the box in (8) at $s_0 = 1/5$. After 15 pairs, upward rounding of the upper corner supplies

$$\bar{r} = \frac{364294600485702452729}{302231454903657293676544}, \qquad \bar{s} = \frac{6297708825960169464219 65}{120892581961462917470 6176}.$$

With $c = 11/20$, the exact strict cap slack is

$$(c - \bar{s}) - \frac{2\bar{r}}{1 - c/\beta} = \frac{830182727008321022239}{60446290980731458735308 80} > 0.$$

To see why this certifies the infinite tail, start each alpha-beta pair below the cap with $r \le \bar{r} < c < \beta$. Alpha cannot increase r; the subsequent beta multiplies it by at most $c/\beta < 1$ as long as the intermediate second coordinate stays below c. Each pair increases s by at most twice its starting r. The geometric total is bounded by $2\bar{r}/(1 - c/\beta) < c - \bar{s}$. A first-exit argument bounds a putative half-pair exit as well and closes the cap induction. Therefore $s < c$ forever and $r \to 0$.

This alternating word satisfies both-symbols-in-every-two-updates, hence both one-sided and two-sided block contracts for every $K \ge 2$. Consequently $G_K = 0$ for all $K \ge 2$ at this critical state. The rational upper start $(6/25, 1/5)$ is beta-positive but has the same alternating zero tail; therefore global switched/constant-beta zero-basin identification also genuinely fails. The bracket, finite iterations and cap are reproduced by the companion certificate code. The finite dyadic computations use the full piecewise maps and round both coordinates outward after each update; isotonicity propagates the resulting enclosures. This isolated certified obstruction does not characterize the maximal good parameter range or the complete behavior below $\alpha = \beta/2$.

9 Singular limits, positioning and scope

Every limit above fixes both parameters and the critical initial point. If $\alpha = \beta$, the modes coincide and critical margins are zero under all words; the positive coefficients for strict pairs do not extend by taking that parameter limit first. As $\alpha \uparrow \beta$, entry into the mixed strip requires $d < \beta - \alpha$ and can be arbitrarily late. As $\beta \to 0$ or $\beta \to 1$, the local radius, constants and admissible initial-state domains change; the coefficient $2\beta/(1-\beta)$ is not a uniform theorem in those limits. No interchange of limits or parameter-uniform remainder is claimed.

The primary antecedents have distinct scopes. In the coordinates $u = d$, $v = -(h + d)r/h$, the beta map is $u' = u + hv$, $v' = v(1 - u/\beta)(1 + hv/(h + u))$. It satisfies analytic case 3 of Cufí-Cabré–Fontich Theorem 2.1 (the 2020 preprint version linked in [1]): $c = h > 0$, $p = 0$, $k = 4$, $l = 2$, $b_l = -1/\beta < 0$. Their local leading coefficients match this independent special-case derivation. That antecedent does not supply the global schedule ordering or obstruction. Automaton-constrained switching [2], Definition 1 and Theorem 2.3, concerns linear multinorm stability; monotone repeated-control safety [3], Theorems 1–2, concerns selected safe policies. Neither result establishes this nonlinear worst-word ordering or these margin coefficients. Historical priority remains unresolved. The result is a focused theorem for the declared model; its scope does not extend automatically to general fuzzy or matrix-algebra systems. Observation margin concerns exact trajectories; no process disturbance or stochastic robustness guarantee is inferred.

10 Reproducible certificates and limitations

The companion audit uses exact symbolic arithmetic and directed rational state bounds. In particular, at $\beta = 1/10$, $\alpha = 1/20$, a positive critical-graph cubic coefficient is allowed; at $\alpha = 1/4$, $\beta = 1/2$ the sufficient-range endpoint is tested. At these and other stated parameter pairs, the exact initial point is specified implicitly as $(\phi_\beta(\beta/3), \beta/3)$. Rational inner/outer certificates bracket its first coordinate, and outward full-map propagation certifies positive period-two margins. The obstruction uses the different exact implicit state specified in its section. These computations corroborate proofs; finite tests do not establish the continuum theorems or historical novelty.

The core proofs above are self-contained. The certificate programs are supplied with the source package; they require Python and SymPy and use exact rational or symbolic arithmetic for certification. The remaining open classification is the maximal good parameter/state region below $\alpha = \beta/2$; no further result on that region is claimed. Observation-margin robustness does not imply robustness to process disturbances.

Research status and AI assistance

This research preprint was developed with generative AI assistance in drafting, proof development, computation, and internal adversarial checking. The internal checks do not constitute external human peer review or formal proof verification. Historical priority remains unverified. The companion source package identifies the reviewed version, executable certificates, and the scope of each check.

References

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