# Integral secondary maps in an observation four-cycle

Technical addendum to Observation Refinement

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**Abstract**

We answer the integral four-cycle realization question in the v0.5 manuscript *Successive Concept Differentiation as Observation Refinement*. For every integer $k \geq 2$, an actual Boolean-observation refinement with exclusive graph exactly $K_{2,2}$ has contractible coarse order complex and torsion-free integral homology in every mixed-core available link, while its only secondary differential is $d^2 : E^2_{4,1} = \mathbb{Z} \to E^2_{2,2} = \mathbb{Z}$, multiplication by $\pm k$. A uniform construction uses $24k+11$ coarse states; a separate projective-plane construction uses 35 states for $k = 2$. A local-anchor modification preserves the entire available-link diagram while contracting the coarse complex. The refined complex has reduced integral homology $\mathbb{Z}/k$ in degree two only. Conversely, for one exclusive $K_{2,2}$ plus isolated exclusive vertices, a surviving source in row zero has a zero or primitive secondary image, including after nontrivial first-page quotients. The constructions use classical Moore spaces and order homotopies. Priority of the exact specialization is not established; the checks and reviews supplied here are internal AI-assisted work, not external peer review.

## 1 Scope and antecedents

This addendum concerns a finite observation-poset model, not a cognitive mechanism. Its precise target is the question headed Integral secondary maps in a four-cycle in Section 19 of the inspected referee-repair draft v0.5, 29 September 2026 [1]. That draft already gives a unit secondary-map example and torsion on larger exclusive stars. The result here imposes exactly one four-cycle and torsion-free available-link homology, and additionally makes the whole coarse complex contractible.

The ingredients are classical: Moore spaces, suspension and pair exact sequences [2], and beat-point/order-homotopy reductions [3]. Available links are instances of the obstruction-complex framework of Chachólski–Jin–Scolamiero–Tombari [4]. Dey–Hirani–Krishnamoorthy [5], especially Section 4.2 and Theorem 5.2, already distinguish absolute torsion-freeness from relative torsion and total unimodularity. General poset filtration spectral sequences are also prior machinery [6]. Our assertion is an explicit realization and a degree-specific obstruction within the observation product, not a new general torsion or spectral-sequence construction. Targeted searches have not resolved historical priority.

## 2 The observation product and its integral interaction complex

All homology has coefficients in $\mathbb{Z}$. A finite poset $Q$ has one-only states $P$, zero-only states $Z$, and split states allowing both Boolean values. The refined poset $Q^\psi$ is the induced available subposet of $Q \times \{0 < 1\}$. Write $s_i = (s, i)$ and put

$$K = \Delta(Q \times 2), \quad L = \Delta Q^\psi, \quad M_0 = P \times \{0\}, \quad M_1 = Z \times \{1\}, \quad U = K[V(K) \setminus M_0] \cup K[V(K) \setminus M_1].$$

For a mixed missing chain $S$ meeting both missing colours, its available link is $\Lambda_S = \mathrm{lk}_K(S)[V(L)]$. Link chains are augmented, with the empty face in degree $-1$. The simplices of $C_*(K,U)$ have the unique basis description $(S,\sigma)$, where $\sigma$ is an available-link face, in degree $|S|+|\sigma|-1$. Orient missing vertices first. Core deletion uses its alternating sign; available-vertex deletion uses $(-1)^{|S|}$ times its augmented boundary. Filtering by $|S|$ gives

$$E^1_{p,q} = \bigoplus_{|S|=p} \widetilde{H}_q(\Lambda_S) \Longrightarrow H_{p+q}(K,U), \qquad d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

This recovers the interaction filtration of [1] from its relative basis.

The exclusive graph has edges $p < z$, for $p \in P, z \in Z$. For $P = \{a,b\}$, $Z = \{x,y\}$ and $a < b < x < y$, the mixed cores are edges $ax, ay, bx, by$, triples $abx, aby, axy, bxy$, and the unique four-core $a_0b_0x_1y_1$. Their augmented chain sums $A, B, V$ give

$$C_m(K,U) = A_{m-2} \oplus B_{m-3} \oplus V_{m-4}, \qquad d(\alpha,\beta,c) = (\partial\alpha + h\beta, -\partial\beta + \kappa c, \partial c). \tag{1}$$

In that triple order $\kappa$ has signs $(-,+,-,+)$, and

$$h(abx) = bx - ax, \quad h(aby) = by - ay, \quad h(axy) = ax - ay, \quad h(bxy) = bx - by,$$

each term meaning its actual chain inclusion. If $\partial c = 0$ and $\partial\beta = \kappa c$, then $d^2[c] = [h\beta]$.

Each $K[V(K) \setminus M_i]$ retracts to a full row by raising the second coordinate to 1 for $i = 0$, or lowering it to 0 for $i = 1$. Its inclusion in $K$ is a homotopy equivalence. The relative Mayer–Vietoris sequence, since the intersection is $L$, gives the degree shift

$$\widetilde{H}_n(\mathrm{Cone}(\Delta\pi)) \cong H_n(K,L) \cong H_{n+1}(K,U), \tag{2}$$

where $\pi : Q^\psi \to Q$ and $K \simeq \Delta Q$ by product projection.

# 3 Local anchors preserve the diagram and contract the coarse poset

Let $R$ be finite, $b$ minimal in $R$, $x$ maximal in $R$, and $b < x$. Start with $Q_0 = \{a\} \oplus R \oplus \{y\}$ and adjoin split states $a', y'$ using only the generating relations

$$a' < b, \qquad x < y'. \tag{3}$$

Take transitive closure and no further comparisons. Make $a, b$ one-only, $x, y$ zero-only, and everything else split. In particular $a \parallel a'$, $y \parallel y'$ and $a' < b < x < y'$. The exclusive graph is exactly $K_{2,2}$.

**Lemma 3.1.** *The complex $\Delta Q$ is contractible. All nine available links and their inclusion maps are literally identical on labelled vertices to those in $\widehat{Q} = \{a, a'\} \oplus R \oplus \{y, y'\}$ with the same profile.*

*Proof.* The strict upper set of $a'$ has minimum $b$, so sending $a'$ to $b$ is an order-preserving retraction pointwise above the identity. Delete $a'$. The strict lower set of $y'$ then has maximum $x$; sending $y'$ to $x$ is a retraction pointwise below the identity. The remaining $Q_0$ has least element $a$. These order homotopies contract $\Delta Q$.

Only anchor comparisons were removed from $\widehat{Q}$. If a mixed core contains $a_0$, neither copy of $a'$ occurs in its link. Otherwise it contains $b_0$, allowing only $a'_0$. Minimality of $b$ means every other

available vertex comparable with $b_0$ lies above it, apart from $a'_0$ itself: the other old lower vertex $a_0$ is unavailable. All needed comparisons from $a'_0$ follow from $a' < b$ and remain. Dually, a core with $y_1$ excludes both copies of $y'$; otherwise it contains $x_1$ and allows only $y'_1$. Maximality of $x$ retains all needed comparisons to this vertex. If both anchors occur, $a'_0 < y'_1$ remains. Nothing else changes, so all simplices and core-deletion inclusions agree. □

These retractions do not preserve availability: $a'$ is split and $b$ is one-only. They do not assert that refinement is neutral.

The remaining link descriptions are

$$\begin{aligned} \Lambda_{aby} &\simeq \Sigma\Delta(R_{>b} \setminus \{x\}), \quad \Lambda_{axy} \simeq \Sigma\Delta(R_{<x} \setminus \{b\}), \\ \Lambda_{abxy} &\simeq \Sigma\Delta(b, x), \qquad\qquad \Lambda_{ay} \simeq \Sigma\Delta(R \setminus \{b, x\}). \end{aligned} \tag{4}$$

To see the $ay$ formula, its lower row is a cone with apex $y_0$ and its upper row a cone with apex $a_1$. The intervening product complex is $\Delta((R \setminus \{b, x\}) \times 2)$. Minimality of $b$ and maximality of $x$ exclude other cross-row chains; the two row cones give its suspension by a homotopy pushout. The triple formulas follow from the same row decomposition. The other five links are cones: $ax, abx$ with apex $y'_1$, $by, bxy$ with apex $a'_0$, and $bx$ with either. The four-core central product interval similarly has two contractible row cones over $(b, x)$, giving its suspension.

## 4 Uniform realization and the integral index

**Theorem 4.1.** *For each $k \geq 2$ there is an actual Boolean-observation realization with $24k + 11$ coarse states, exclusive graph exactly $K_{2,2}$, contractible $\Delta Q$ and torsion-free integral homology in every available mixed-core link. Its only nonzero first-page terms are $E^1_{4,1} = \mathbb{Z}$ and $E^1_{2,2} = \mathbb{Z}$. The map $d^2 : E^2_{4,1} \to E^2_{2,2}$ is multiplication by $\pm k$. Moreover*

$$\widetilde{H}_i(\Delta Q^\psi) = \begin{cases} \mathbb{Z}/k & i = 2, \\ 0 & i \neq 2, \end{cases} \qquad \widetilde{H}_i(\mathrm{Cone}(\Delta\pi)) = \begin{cases} \mathbb{Z}/k & i = 3, \\ 0 & i \neq 3. \end{cases}$$

*Proof.* Put $n = 3k$. On distinct vertices $v_0, v_1, v_2, u_0, \ldots, u_{n-1}, w$, define $F_k$ by these facets for $j = 0, \ldots, n-1$:

$$\{u_j, u_{j+1}, v_{j+1 \bmod 3}\}, \quad \{u_j, v_{j \bmod 3}, v_{j+1 \bmod 3}\}, \quad \{w, u_j, u_{j+1}\}, \tag{5}$$

where $u$ indices are modulo $n$. The first two families triangulate the mapping cylinder of the $k$-fold covering $C_{3k} \to C_3$, given by $u_j \mapsto v_{j \bmod 3}$; the third cones off its domain. This is a simplicial Moore-space model, with cellular attaching boundary $[k]$, so $H_1(F_k) = \mathbb{Z}/k$, $H_2(F_k) = 0$. Its face numbers are $(3k+4, 12k+3, 9k)$. Let $R$ be its nonempty-face poset, $b = \{w\}$, $x = \{w, u_0, u_1\}$, and use (3). There are $24k + 7$ faces and four anchors.

The link of $w$ is $C_{3k}$. Identify each face above $b$ with that face minus $w$. Deleting $x$ breaks the barycentric circle of this link into a path. Thus $\Delta(R_{>b} \setminus \{x\})$ is contractible. The triangle boundary $\Delta(R_{<x} \setminus \{b\})$ is also a path, and $(b, x)$ consists of the two incomparable edges $wu_0, wu_1$. Formula (4) gives contractible $aby, axy$ and a four-core hexagon homotopy equivalent to $S^1$.

Set $C = R \setminus \{b, x\}$. In $\Delta R$, the vertex $b$ is the original point $w$. A simplicial vertex puncture retracts onto the induced vertex deletion by normalizing the other barycentric coordinates. The punctured cone at $w$ retracts onto its base, fixing it, so punctured $F_k$ retracts to the covering cylinder and is homotopy equivalent to $S^1$. After deleting $b$, the link of $x$ is the path above. Restoring $x$

attaches a cone over a contractible subcomplex, hence is a homotopy equivalence. Thus $\Delta C \simeq S^1$ and $\Lambda_{ay} \simeq S^2$. Along with the five cone links this proves every first-page assertion.

For the integral index let $T = \Delta((a_0, y_1)_{Q\times 2})$ and let $Y = T \setminus \{b_0, x_1\} = \Lambda_{ay}$. Neither auxiliary anchor occurs in $T$. The row decomposition gives $T \simeq \Sigma\Delta R$, $Y \simeq S^2$. Let $J$ be the union of the complexes obtained by restoring $b_0$ or $x_1$ separately to $Y$. The relative chains of $(J, Y)$ are the direct sum of shifted augmented triple-link chains $aby, axy$, so are integrally acyclic. The chains of $(T, J)$ are augmented four-core chains shifted by two. Therefore

$$H_3(T, Y) \xrightarrow{\cong} H_3(T, J) = H_1(\Lambda_{abxy}) = \mathbb{Z}, \qquad H_2(T, Y) = 0.$$

This integral isomorphism fixes a primitive source. Since $H_3(T) = 0$, $H_2(T) = \mathbb{Z}/k$, $H_2(Y) = \mathbb{Z}$ and $H_1(Y) = 0$, the pair sequence reduces to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \longrightarrow \mathbb{Z}/k \longrightarrow 0. \tag{6}$$

Hence $\delta = \pm k$. Its chain construction takes the two single-missing-vertex fillings in the actual triple links $aby, axy$ and their signed boundary in $Y$. This is exactly the $ay$ coordinate of $h\beta$ in (1). The other edge targets and remaining triples have zero homology, so this is the full secondary map up to orientation. Only $\mathbb{Z}/k$ at $(2, 2)$ survives; no later differential or filtration extension is possible. Equation (2) gives cofiber homology, and the mapping-cone sequence with $\Delta Q$ contractible gives refined homology. □

**Proposition 4.2.** *These profiles are realized by Boolean observations on* $2|Q| - 4$ *worlds.*

*Proof.* Use a world $(s, t)$ for each allowed bit at $s$. For each $q \in Q$, set $\varphi_q(s, t) = 1$ iff $q \le s$. Old signatures are principal-downset indicators: equality means the same $s$, and coordinatewise comparison is exactly the order on $Q$. Adding $\psi(s, t) = t$ produces precisely the induced available product order. No isotonicity of $\psi$ on $Q$ is assumed. □

## 5 A separate 35-state realization for k=2

Take the six-vertex triangulation of $\mathbb{RP}^2$ with facets

$$012, 013, 024, 035, 045, 125, 134, 145, 234, 235.$$

This seed already occurs in the star example of [1]. Its face poset has $6 + 15 + 10 = 31$ states. Set $b = \{0\}$, $x = \{0, 1, 2\}$ and use the four anchors of (3). There are 35 coarse states and 66 worlds. This is different from the uniform $k = 2$ construction, which has 59 coarse states.

The vertex link at 0 is a circle, its $x$ deletion a path, and $(b, x) = \{01, 02\}$ is an antichain. The punctured projective plane retracts to a Moebius band and then a circle. Deleting $x$ afterward attaches/removes a cone over the same contractible path. The preceding link and integral-index proofs therefore apply with $k = 2$: seven links are contractible, $\Lambda_{abxy} \simeq S^1$ and $\Lambda_{ay} \simeq S^2$. With $e = 01, f = 02$, a primitive source is

$$c = [b_1, e_1] - [b_1, f_1] - [e_0, e_1] + [e_0, x_0] - [f_0, x_0] + [f_0, f_1].$$

The stored four fillings have boundaries $(-, +, -, +)c$. Their $ay$ image is $-2g + \partial t$, with primitive target $g$ and an integral cocycle $\eta$ satisfying $\eta(g) = 1$. Complete chains $g, t, \eta$ are supplied in the certificate package. Reversing the source gives multiplication by $+2$.

## 6 The lowest source row is zero or primitive

A subgroup generated by a nonzero vector of a free abelian group is primitive if the quotient by it is torsion-free. The next theorem does not require torsion-free higher link homology or vanishing $d^1$.

**Theorem 6.1.** *Suppose the entire exclusive graph is one $K_{2,2}$ and optional isolated exclusive vertices. If $E^2_{4,0} \neq 0$, then $E^2_{4,0} = \mathbb{Z}$, $E^2_{2,1}$ is free abelian, and $d^2 : E^2_{4,0} \to E^2_{2,1}$ has zero or primitive image. In particular, for rank-one target its scalar is $0, 1$ or $-1$.*

*Proof.* A four-core requires comparable same-colour pairs; label them $a < b < x < y$. A missing copy outside this component cannot enter a mixed-core link: it would be comparable with a missing vertex of the opposite colour in the direction creating an additional exclusive edge. Hence the four-core available link is its full product link. It factors as a join of five interval complexes:

$$\Delta Q_{<a}, \quad \Delta(a,b), \quad \Delta((b_0,x_1)_{Q\times 2}), \quad \Delta(x,y), \quad \Delta Q_{>y}.$$

The middle factor contains $b_1, x_0$. If any other factor is nonempty, this join is connected. A state $r \in (b,x)$ connects the middle factor by $b_1 < r_1$, $r_0 < r_1$, $r_0 < x_0$; all its other vertices connect to these caps. Thus a nonzero reduced $H_0$ forces

$$Q_{<a} = (a,b) = (b,x) = (x,y) = Q_{>y} = \varnothing.$$

Consequently $a$ is minimal, $y$ maximal, the three marked comparisons are covers, and $\Lambda_{abxy} = \{b_1, x_0\}$. Its primitive source is $c = [b_1] - [x_0]$. There is no column five. Its map into triple reduced $H_0$ has free target, so a nonzero kernel from $\mathbb{Z}$ is all of $\mathbb{Z}$. Survival means $b_1, x_0$ connect in every triple.

Use the following subposets of the whole $Q$:

$$I = Q_{<b} \setminus \{a\}, \quad T = Q_{>x} \setminus \{y\}, \quad D = (a,x) \setminus \{b\}, \quad B = (b,y) \setminus \{x\}, \quad C = (a,y) \setminus \{b,x\}.$$

All states of $B, C, D$ split; an extra exclusive state in these intervals would create an extra exclusive edge. Every 0-copy in $I$ and 1-copy in $T$ is available, including at isolated exclusive states. Product-interval factorization gives

$$\Lambda_{abx} = S^0 * \Delta T, \qquad \Lambda_{bxy} = \Delta I * S^0, \qquad \Lambda_{axy} \simeq \Sigma\Delta D, \quad \Lambda_{aby} \simeq \Sigma\Delta B, \quad \Lambda_{ay} \simeq \Sigma\Delta C.$$

For example, in $axy$ the retained $b_1$ attaches only at $a_1$, since the marked comparisons are covers. Removing this ear leaves the two-row suspension model on $D$. The $aby$ description is dual. These descriptions also show $I, T, D, B$ must be nonempty for source survival. The remaining edge links factor as

$$\Lambda_{ax} = \Lambda_{axy} * \Delta T, \quad \Lambda_{by} = \Delta I * \Lambda_{aby}, \quad \Lambda_{bx} = \Delta I * S^0 * \Delta T.$$

A join of two nonempty complexes is connected; these formulas and the preceding suspension descriptions imply their $H_1$ groups vanish. Thus the complete target, including every $d^1$ contribution, is

$$E^2_{2,1} = H_1(\Lambda_{ay}) / \big(\operatorname{im} H_1(\Lambda_{aby}) + \operatorname{im} H_1(\Lambda_{axy})\big). \tag{7}$$

To determine the actual inclusion lattices integrally, let $\Omega = \pi_0(\Delta C)$. On an oriented bit-changing edge of $ay$ set $\Psi([s_0,t_1]) = e_{[s]} \in \mathbb{Z}[\Omega]$, and on same-row edges set $\Psi = 0$. Every crossing has $s, t \in C$, in the same component. The vertices $b_1, x_0, a_1, y_0$ cannot provide any additional crossings under the cover conditions. On a triangle of bit pattern 001 or 011 the crossing terms

cancel; the coordinate sum on a cycle is zero because bit height is a potential. The row-cone decomposition therefore induces an integral isomorphism

$$H_1(\Lambda_{ay}) \xrightarrow{\Psi} \widetilde{H}_0(\Delta C) = \ker(\mathbb{Z}[\Omega] \xrightarrow{\text{sum}} \mathbb{Z}).$$

One can see the normalization by collapsing the two contractible row cones: each component contributes one connecting edge. Restriction to $axy, aby$ uses exactly the crossing states in $D, B$, respectively. Hence the two images in (7) are generated by all $e_i - e_j$ for components meeting $D$, and separately all such differences for components meeting $B$.

Let $\sim$ identify all components meeting $D$, and separately those meeting $B$. Then $E^2_{2,1}$ is the augmentation kernel of $\mathbb{Z}[\Omega/\sim]$, in particular free. Choose $d \in D, t \in B$. The paths

$$x_0 - d_0 - d_1 - a_1 - b_1, \qquad x_0 - y_0 - t_0 - t_1 - b_1$$

from $x_0$ to $b_1$ fill the source in $axy$ and $aby$. The other two fillings exist through any 1-copy in $T$ and any 0-copy in $I$. The signed $ay$ image is the difference of the two displayed paths, so maps to $e_{[d]} - e_{[t]}$ in this quotient, up to overall sign. If these two component classes merge, it is zero. Otherwise an integer functional equal to 1 on the $D$-class and 0 elsewhere annihilates both $d^1$ lattices and evaluates to 1 on the image. The image is then primitive, proving the theorem. □

**Corollary 6.2.** *For the graph scope of Theorem 6.1, row $q = 1$ is the first possible source row for a nonunit rank-one secondary map.*

*Proof.* The four-core always contains $b_1, x_0$, so reduced $H_{-1}$ cannot contribute. Theorem 6.1 excludes a nonunit scalar in row zero, and Theorem 4.1 realizes every $k \geq 2$ in row one. □

This is a source-degree statement, not a minimum-state assertion. It does not cover a $K_{2,2}$ subgraph inside a larger connected exclusive component, or several nontrivial components. No classification under those broader hypotheses is asserted.

# 7 Certificates, checks and limits

The supplied evidence distinguishes general proofs from finite tests. The construction/index proofs work for arbitrary $k$; finite cases are implementation checks, not extrapolations. Complete model, link-boundary, relative-boundary and signed-filling files are included for the 35-state example and uniform $k = 2, 3, 4, 5$. In the saved bases their scalars are $-2, -2, 3, 4, 5$. For the 35-state relative complex the ranks in degrees $1, 2, 3, 4, 5$ are $4, 110, 514, 778, 370$; boundary ranks from degree two upward are $4, 106, 408, 370$. The only nonunit Smith factor is 2 in degree-five boundary. The refined complex has 66 vertices and 1464 augmented simplices, and reduces integrally to [2] from degree three to degree two.

Four additional low-degree examples check zero and unit maps, a nontrivial $d^1$ quotient with isolated exclusive vertices, and a higher-rank primitive image. Their whole diagrams and integral lattices are retained. The fresh JavaScript checks rebuild the row-one models from facet and anchor generators, and independently reconstruct all tested product links, chain boundaries and integral quotients using exact integer arithmetic; historical Python/SymPy Smith checks provide another route. The public review summary identifies which computations were performed in the current closeout and which are retained evidence. AI proof and hostile reviews are not external human peer review or formal proof-assistant certification.

The result concerns homology, without asserting a homotopy classification of the refined complex. Coarse contractibility does not force every interval to be torsion-free: the interval $(a, y)$

here is the Moore-space face poset. Minimum counts, historical priority and larger-graph classifications remain open. No physical validation, consciousness criterion or architectural advantage follows. Specialist human review remains recommended before treating this exact specialization as externally vetted.

## References


[1] Brian Theory, *Successive Concept Differentiation as Observation Refinement: Carrier Splitting, Interaction-Core Homology, and Descent*, referee-repair draft v0.5, 29 September 2026, Sections 7–9 and 19. `https://www.b-theory.com/manuscripts-html/observation-refinement/`. The dependency is the fixed source archive identified in the source ledger.

[2] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002, Chapter 2, especially Example 2.40 (Moore spaces) and reduced Mayer–Vietoris. `https://pi.math.cornell.edu/~hatcher/AT/ATch2.pdf`.

[3] J. A. Barmak and E. G. Minian, Simple homotopy types and finite spaces, *Adv. Math.* 218 (2008), 87–104. doi:10.1016/j.aim.2007.11.019. Sections 2.1–2.3. `https://arxiv.org/abs/math/0611158`.

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[5] T. K. Dey, A. N. Hirani and B. Krishnamoorthy, Optimal homologous cycles, total unimodularity, and linear programming, *SIAM J. Comput.* 40 (2011), 1026–1044. doi:10.1137/100800245. Section 4.2 and Theorem 5.2. `https://arxiv.org/html/1001.0338v3`.

[6] N. Cianci and M. Ottina, A new spectral sequence for homology of posets, *Topology Appl.* 217 (2017), 1–19. doi:10.1016/j.topol.2016.12.001.